CHAPTER 1

MATHEMATICS

ARITHMETIC

100. Definition

Arithmetic is that branch of mathematics dealing with computation by numbers. The principal processes involved are addition, subtraction, multiplication, and division. A number consisting of a single symbol (1, 2, 3, etc.) is **a digit**. Any number that can be stated or indicated, however large or small, is called a **finite number**; one too large to be stated or indicated is called an **infinite number**; and one too small to be stated or indicated is called an **infinitesimal number**.

The **sign** of a number is the indication of whether it is positive (+) or negative (-). This may sometimes be indicated in another way. Thus, latitude is usually indicated as *north* (N) or *south* (S), but if north is considered positive, south is then negative with respect to north. In navigation, the north or south designation of latitude and declination is often called the "name" of the latitude or declination. A **positive number** is one having a positive sign (+); a **negative number** is one having a negative sign (-). The **absolute value** of a number is that number without regard to sign. Thus, the absolute value of both (+) 8 and (-) 8 is 8. Generally, a number without a sign can be considered positive.

101. Significant Digits

Significant digits are those digits of a number which have a significance. Zeros at the left of the number and sometimes those at the right are excluded. Thus, 1,325, 1,001, 1.408, 0.00005926, 625.0, and 0.4009 have four significant digits each. But in the number 186,000 there may be three, four, five, or six significant digits depending upon the accuracy with which the number has been determined. If the quantity has only been determined to the nearest thousand then there are three significant digits, the zeros at the right not being counted. If the number has been determined to the nearest one hundred, there are four significant digits, the first zero at the right being counted. If the number has been determined to the nearest ten, there are five significant digits, the first two zeros on the right being counted. If the quantity has been determined to the nearest unit, there are six significant digits, the three zeros at the right being counted.

This ambiguity is sometimes avoided by expressing numbers in powers of 10. Thus, $18.6\ \ 10^4$ (18.6 x 10,000) indicates accuracy to the nearest thousand, $18.60\ \ 10^4$ to the nearest hundred, $18.600\ \ 10^4$ to the nearest ten, and $18.6000\ \ 10^4$ to the nearest unit. The position of the decimal is not important if the correct power of 10 is given. For example, $18.6\ \ 10^4$ is the same as $1.86\ \ 10^6$, $186\ \ 10^3$, etc. The small number above and to the right of 10 (the **exponent**) indicates the number of places the decimal point is to be moved to the right. If the exponent is negative, it indicates a reciprocal, and the decimal point is moved to the left. Thus, $1.86\ \ 10^{-6}$ is the same as 0.00000186. This system is called **scientific notation**.

102. Expressing Numbers

In navigation, fractions are usually expressed as decimals. Thus, $1/4$ is expressed as 0.25 and $1/3$ as 0.33. To determine the decimal equivalent of a fraction, divide the **numerator** (the number above the line) by the **denominator** (the number below the line). When a decimal is less than 1, as in the examples above, it is good practice to show the zero at the left of the decimal point (0.25, not .25).

A number should not be expressed using more significant digits than justified. The *implied* accuracy of a decimal is indicated by the number of digits shown to the right of the decimal point. Thus, the expression "14 miles" implies accuracy to the nearest whole mile, or any value between 13.5 and 14.5 miles. The expression "14.0 miles" implies accuracy of a tenth of a mile, or any value between 13.95 and 14.05 miles.

A quantity may be expressed to a greater implied accuracy than is justified by the accuracy of the information from which the quantity is derived. For instance, if a ship steams 1 mile in 3^m21^s, its speed is $60^m \div 3^m21^s = 60 \div 3.35 = 17.910447761194$ knots, approximately. The division can be carried to as many places as desired, but if the time is measured only to the nearest second, the speed is accurate only to one decimal place in this example, because an error of 0.5 second introduces an error of more than 0.05 knot in the speed. Hence, the additional places are meaningless and possibly misleading, unless more accurate time is available. In general, it is not good practice to state a quantity to imply accuracy greater than what is justified. However, in marine navigation the accuracy of information is often unknown, and it is customary to give positions as if they were accurate to 0.1' of latitude and longitude, although they *may* not be accurate even to the nearest whole minute.

If there are no more significant digits, regardless of how far a computation is carried, this may be indicated by use of the word “exactly.” Thus, 12÷4=3 exactly and 1 nautical mile = 1,852 meters exactly; but 12÷7=1.7 approximately, the word “approximately” indicating that additional decimal places might be computed. Another way of indicating an approximate relationship is by placing a positive or negative sign after the number. Thus, 12÷7=1.7+, and 11÷7=1.6-. This system has the advantage of showing whether the approximation is too great or too small.

In any arithmetical computation the answer is no more accurate than the least accurate value used. Thus, if it is desired to add 16.4 and 1.88, the answer might be given as 18.28, but since the first term might be anything from 16.35 to 16.45; the answer is anything from 18.23 to 18.33. Hence, to retain the second decimal place in the answer is to give a false indication of accuracy, for the number 18.28 indicates a value between 18.275 and 18.285. However, additional places are sometimes retained until the end of a computation to avoid an accumulation of small errors due to rounding off. In marine navigation it is customary to give most values to an accuracy of 0.1, even though some uncertainty may exist as to the accuracy of the last place. Examples are the dip and refraction corrections of sextant altitudes.

In general, a value obtained by interpolation in a table should not be expressed to more decimal places than given in the table.

Unless all numbers are exact, doubt exists as to the accuracy of the last digit in a computation. Thus, 12.3+9.4+4.6=26.3. But if the three terms to be added have been rounded off from 12.26, 9.38, and 4.57, the correct answer is 26.2, obtained by rounding off the answer of 26.21 found by retaining the second decimal place until the end. It is good practice to work with one more place than needed in the answer, when the information is available. In computations involving a large number of terms, or if greater accuracy is desired, it is sometimes advisable to retain two or more additional places until the end.

103. Rounding Off

In **rounding off** numbers to the number of places desired, one should take the nearest value. T the number 6.5049 is rounded to 6.505, 6.50, 6.5, or 7, depending upon the number of places desired. If the number to be rounded off ends in 5, the nearer even number is taken. Thus, 1.55 and 1.65 are both rounded to 1.6. Likewise, 12.750 is rounded to 12.8 if only one decimal place is desired. However, 12.749 is rounded to 12.7. That is, 12.749 is not first rounded to 12.75 and then to 12.8, but the entire number is rounded in one operation. When a number ends in 5, the computation can sometimes be carried to additional places to determine whether the correct value is more or less than 5.

104. Reciprocals

The **reciprocal** of a number is 1 divided by that number. The reciprocal of a fraction is obtained by interchanging the numerator and denominator. Thus, the reciprocal of $3/5$ is $5/3$. A whole number may be considered a fraction with 1 as the denominator. Thus, 54 is the same as $54/1$, and its reciprocal is $1/54$. Division by a number produces the same result as multiplying by its reciprocal, or vice versa. Thus, $12\div2=12\times1/2=6$, and $12\times2=12\div1/2=24$.

105. Addition

When two or more numbers are to be added, it is generally most convenient to write them in a column, with the decimal points in line. Thus, if 31.2, 0.8874, and 168.14 are to be added, this may be indicated by means of the addition sign (+): 31.2+0.8874+168.14=200.2. But the addition can be performed more conveniently by arranging the numbers as follows:

$$\begin{array}{r} 31.2 \\ 0.8874 \\ 168.14 \\ \hline 200.2 \end{array}$$

The answer is given only to the first decimal place, because the answer is no more accurate than the least precise number among those to be added, as indicated previously. Often it is preferable to state all numbers in a problem to the same precision before starting the addition, although this may introduce a small error:

$$\begin{array}{r} 31.2 \\ 0.9 \\ 168.1 \\ \hline 200.2 \end{array}$$

If there are no decimals, the last digit to the right is aligned:

$$\begin{array}{r} 166 \\ 2 \\ 96,758 \\ \hline 96,926 \end{array}$$

Numbers to be added should be given to the same absolute accuracy, when available, to avoid a false impression of accuracy in the result. Consider the following:

$$\begin{array}{r} 186,000 \\ 71,832 \\ 9,614 \\ 728 \\ \hline 268,174 \end{array}$$

The answer would imply accuracy to six places. If the first number given is accurate to only three places, or to the nearest 1,000, the answer is not more accurate, and hence the answer should be given as 268,000. Approximately the same answer would be obtained by rounding off at the start:

$$\begin{array}{r} 186,000 \\ 72,000 \\ 10,000 \\ 1,000 \\ \hline 269,000 \end{array}$$

If numbers are **added arithmetically**, their absolute values are added without regard to signs; but if they are **added algebraically**, due regard is given to signs. If two numbers to be added algebraically have the same sign, their absolute values are added and given their common sign. If two numbers to be added algebraically have unlike signs, the smaller absolute value is subtracted from the larger, and the sign of the value having the larger absolute value is given to the result. Thus, if +8 and -7 are added arithmetically, the answer is 15, but if they are added algebraically, the answer is + 1.

An answer obtained by addition is called a **sum**.

106. Subtraction

Subtraction is the inverse of addition. Stated differently, the *addition* of a *negative* number is the same as the *subtraction* of a *positive* number. That is, if a number is to be subtracted from another, the sign (+ or -) of the **subtrahend** (the number to be subtracted) is reversed and the result added algebraically to the **minuend** (the number from which the subtrahend is to be subtracted). Thus, 6-4=2. This may be written +6-(+4)=+2, which yields the same result as +6+ (-4). For solution, larger numbers are often conveniently arranged in a column with decimal points in a vertical column, as in addition. Thus, 3,728.41-1,861.16 may be written:

(+)3,728.41
(+)1,861.16 (subtract)
(+)1,867.25

This is the same as:

(+)3,728.41
(-)1,861.16 (add algebraically)
(+)1,867.25

The rule of sign reversal applies likewise to negative numbers. Thus, if -3 is to be *subtracted* from +5, this may be written +5- (-3) =5+3=8. In the algebraic addition of two numbers of opposite sign (numerical subtraction), the smaller number is subtracted from the larger and the result is given the sign of the larger number. Thus, +7 -4= +3, and -7 +4= -3, which is the same as +4 -7=-3.

In navigation, numbers to be numerically subtracted are usually marked (-), and those to be numerically added are marked (+) or the sign is not indicated. However, when a sign is part of a designation, and the reverse process is to be used, the word "reversed" (rev.) is written after the number. Thus, if GMT is known and ZT in the (+) 5 zone is to be found (by subtraction), the problem may be written:

GMT	1754	
ZD	(+)5	(rev.)
ZT	1254	

The symbol ~ indicates that an absolute difference is required without regard to sign of the answer. Thus, 28~13=15, and 13~28=15. In both of these solutions 13 and 28 are positive and 15 is an absolute value without sign. If the signs or names of both numbers are the same, either positive or negative, the smaller is subtracted from the larger, but if they are of opposite sign or name, they are numerically added. Thus, (+)16~(+)21=5 and (-)16~(-)21=5, but (+)16~ (-)21=37 and (-)16~(+)21 =37. Similarly, the difference of latitude between 15°N and 20°N, or between 15°S and 20°S, is 5°, but the difference of latitude between 15°N and 20°S, or between15°S and 20°N, is 35°. If motion from one latitude to another is involved, the difference may be given a sign to indicate the direction of travel, or the location of one place with respect to another. Thus, if B is 50 miles west of A, and C is 125 miles west of A, B and C are 75 miles apart regardless of the direction of travel. However, B is 75 miles east of C, and C is 75 miles west of B. When direction is indicated, an algebraic difference is given, rather than an absolute difference, and the symbol ~ is not appropriate.

It is sometimes desirable to consider all addition and subtraction problems as addition, with negative signs (-) given before those numbers to be subtracted; so that there can be no question of which process is intended. The words "add" and "subtract" may be used instead of signs. In navigation, "names" (usually north, south, east, and west) are often used, and the relationship involved in a certain problem may need to be understood to determine whether to add or subtract. Thus, LHA=GHA-λ (west) and LHA=GHA+λ (east). This is the same as saying LHA=GHA-λ if west longitude is considered positive, for in this case, LHA=GHA-(-λ) or LHA=GHA+λ in east longitude, the same as before.

If numbers are **subtracted arithmetically**, they are subtracted without regard to sign; but if they are **subtracted algebraically**, positive (+) numbers are *subtracted* and negative (-) numbers are *added*.

An answer obtained by subtraction is called a **difference**.

107. Multiplication

Multiplication may be indicated by the multiplication sign (×), as 154×28=4,312. For solution, the problem is conveniently arranged thus:

```
  154
(x)28
 1232
 308
 4312.
```

Either number may be given first, but it is generally more convenient to perform the multiplication if the larger number is placed on top, as shown. In this problem, 154 is first multiplied by 8 and then by 2. The second answer is placed under the first, but set one **place** to the left, so that the right-hand digit is directly below the 2 of the multiplier. These steps might be reversed, multiplication by 2 being performed first. This procedure is sometimes used in estimating.

When one number is placed below another for multiplication, as shown above, it is usually best to align the right-hand digits without regard for the position of the decimal point. The number of decimal places in the answer is the sum of the decimal places in the **multiplicand** (the number to be multiplied) and the **multiplier** (the second number):

```
    163.27
(x) 263.9
    146943
    48981
   97962
  32654
  43086.953
```

However, when a number ends in one or more zeros, these may be ignored until the end and then added on to the number:

```
  1924
 (x)1800
 15392
 1924
 3463200
```

This is also true if both multiplicand and multiplier end in zeros:

```
  1924000
 (x)1800
15392
1924
3463200000
```

When negative values are to be multiplied, the sign of the answer is positive if an *even* number of negative signs appear, and negative if there are an *odd* number. Thus, 2×3=6, 2×(-3)=-6, -2×3=-6, -2× (-3)=(+)6. Also, 2×3×8× (-2)×5=-480, 2×(-3)×8×(-2)×5=480, 2×(-3)×(-8)×(-2) ×5=-480, 2×(-3)×(-8)×(-2)×(-5)=480, and (-2)×(-3)×(-8)×(-2)×(-5)=-480.

An answer obtained by multiplication is called a **product**. Any number multiplied by 1 is the number itself. Thus, 125×1=125. Any number multiplied by 0 is 0. Thus, 125×0=0 and l×0=0.

To multiply a number by itself is to **square** the number. This may be indicated by the **exponent** 2 placed to the right of the number and above the line as a **superior**. Thus, 15×15 may be written 15^2. Similarly, 15×I5×I5=15^3, and 15×15×15×15=15^4, etc. The exponent (2, 3, 4, etc.) indicates the **power** to which a number is to be **raised**, or how many times the number is to be used in multiplication. The expression 15^2 is usually read "15 squared", 15^3 is read "15 cubed" or "15 to the third power," 15^4 (or higher power) is read "15 to the fourth (or higher) power." The answer obtained by **raising to a power** is called the "square," "cube" etc., or the ... "power" of the number. Thus, 225 is the "square of 15", 3,375 is the "cube of 15" or the "third power of 15," etc. The zero power of any number except zero (if zero is considered a number) is 1. The zero power of zero is zero. Thus, 15^0=1 and 0^0=0.

Parentheses may be used to eliminate doubt as to what part of an expression is to be raised to a power. Thus, -32 may mean either -(3×3)=-9 or -3×-3=(+)9. To remove the ambiguity, the expression may be written -(3)2 if the first meaning is intended, and $-(3)^2$ if the second meaning is intended.

108. Division

Division is the inverse of multiplication. It may be indicated by the division sign (÷), as 376÷21=18 approximately; or by placing the number to be divided, called the **dividend** (376), over the other number, called the **divisor** (21), as $\frac{376}{21}$=18 approximately. The expression $\frac{376}{21}$ may be written 376/21 with the same meaning. Such a problem is conveniently arranged for solution as follows:

```
     17
  21|376
     21
     166
     147
      19.
```

Since the **remainder** is 19, or more than half of the divisor (21), the answer is 18 to the nearest whole number.

An answer obtained by division is called a **quotient**. Any number divided by 1 is the number itself. Thus, 65÷1=65. A number cannot be divided by 0.

If the numbers involved are accurate only to the number of places given, the answer should not be carried to additional places. However, if the numbers are exact, the answer might be carried to as many decimal places as desired. Thus, 374÷21 =17.809523809523809523809523. . . When a series of digits repeat themselves with the same remainder, as 809523 (with remainder 17) in the example given above, an exact answer will not be obtained regardless of the number of places to which the division is carried. The series of dots (...) indicates a **repeating decimal**. In a non-repeating decimal, a plus sign (+) may be given to indicate a remainder, and a minus sign (-) to indicate that the last digit has been rounded to the next higher value. Thus, 18.68761 may be written 18.6876+ or 18.688-. If the last digit given is rounded off, the word "approximately" may be used instead of dots or a plus or minus sign.

If the divisor is a whole number, the decimal point in the quotient is directly above that of the dividend when the work form shown above is used. Thus, in the example given above, if the dividend had been 37.6 instead of 376, the quotient would have been 1.8 approximately. If the divisor is a decimal, both it and the dividend are multiplied by the power of 10 having an exponent equal to the number of decimal places in the divisor, and the division is then carried out as explained above. Thus, if there are two decimal places in the divisor, both divisor and dividend are multiplied by $10^2 = 100$. This is done by moving the decimal to the right until the divisor is a whole number. If necessary, zeros are added to the dividend. Thus, if 3.7 is to be divided by 2.11, both quantities are first multiplied by 10^2, and 370 is divided by 211. This is usually performed as follows:

```
         1.75
2.11 |3.7000
      211
      1590
      1477
        1130
        1055
          75.
```

If *both* the dividend and divisor are positive, or if *both* are negative, the quotient is positive; but if *either* is negative, the quotient is negative. Thus, 6÷3=2, (- 6)÷(-3)=+2, (-6)÷3=-2, and 6÷(-3)=-2.

The **square root** of a number is that number which, multiplied by itself, equals the given number. Thus, $15 \times 15 = 15^2 = 225$, and $\sqrt{225} = 225^{1/2} = 15$. The square root symbol $\sqrt{\ }$ is called the **radical sign**, or the exponent ½ indicates square root. Also, $\sqrt[3]{\ }$ or 1/3 as an exponent, indicates **cube root.** Fourth, fifth, or any root is indicated similarly, using the appropriate number. Nearly any arithmetic book explains the process of extracting roots, but this process is most easily performed by table, logarithms, or calculator. If no other means are available, it can be done by trial and error. The process of finding a root of a number is called **extracting a root.**

109. Logarithms

Though rarely used today, **logarithms** ("logs") provide an easy way to multiply, divide, raise numbers to powers, and extract roots. The logarithm of a number is the power to which a fixed number, called the base, must be raised to produce the value to which the logarithm corresponds. The base of **common logarithm**, (given in Tables 1 and 3) is 10. Hence, since $10^{1.8} = 63$ approximately, 1.8 is the logarithm, approximately, of 63 to the base 10. In table 1 logarithms of numbers are given to five decimal **places**. This is sufficient for most purposes of the navigator. For greater precision, a table having additional places should be used. In general, the number of *significant digits* which are correct in an answer obtained by logarithms is the same as the number of *places* in the logarithms used.

A logarithm is composed of two parts. That part to the left of the decimal point is called the **characteristic**. That part to the right of the decimal point is called the **mantissa**. The principal advantage of using 10 as the base is that any given combination of digits has the same mantissa regardless of the position of the decimal point. Hence, only the mantissa is given in the main tabulation of table 1. Thus, the logarithm (mantissa) of 2,374 is given as 37548. This is correct for 2,374,000,000; 2,374; 23.74; 2.374; 0.2374; 0.000002374; or for any other position of the decimal point.

The position of the decimal point determines the characteristic, which is not affected by the actual digits involved. The characteristic of a whole number is one less than the number of digits. The characteristic of a **mixed decimal** (one greater than 1) is one less than the number of digits to the left of the decimal point. Thus, in the example given above, the characteristic of the logarithm of 2,374,000,000 is 9; that of 2,374 is 3; that of 23.74 is 1; and that of 2.374 is 0. The complete logarithms of these numbers are:

log 2,374,000,000=9.37548
log 2,374 =3.37548
log 23.74 =1.37548
log 2.374 =0.37548

Since the mantissa of the logarithm of any multiple of ten is zero, the main table starts with 1,000. This can be considered 100, 10, 1, etc. Since the mantissa of these logarithms is zero, the logarithms consist of the characteristic only, and are whole numbers. Hence, the logarithm of 1 is 0 (0.00000), that of 10 is 1 (1.00000), that of 100 is 2 (2.00000), that of 1,000 is 3 (3.00000), etc.

The characteristic of the logarithm of a number less than 1 is negative. However, it is usually more conveniently indicated in a positive form, as follows: the characteristic is found by subtracting the number of zeros immediately to the right of the decimal point from 9 (or 19, 29, etc.) and following this by -10 (or -20, -30, etc.). Thus, the characteristic of the logarithm of 0.2374 is 9-10; that of 0.000002374 is 4-10; and that of 0:000000000002374 is 8-20. The complete logarithms of these numbers are:

log 2.374 =9.37548 - 10
log 0.000002374 =4.37548 - 10
log 0.000000000002374=8.37548 - 20

When there is no question of the meaning, the -10 may be omitted. This is usually done when using logarithms of trigonometric functions, as shown in table 3. Thus, if there is no reasonable possibility of confusion, the logarithm of 0.2374 may be written 9.37548.

Occasionally, the logarithm of a number less than 1 is shown by giving the negative characteristic with a minus sign above it (since only the characteristic is negative, the mantissa being positive). Thus, the logarithms of the numbers given above might be shown thus:

$$\log 0.2374 = \bar{1}.37548$$
$$\log 0.000002374 = \bar{6}.37548$$
$$\log 0.000000000002374 = \overline{12}.37548$$

In each case, the negative characteristic is one more than the number of zeros immediately to the right of the decimal point.

There is no real logarithm of 0, since there is no *finite* power to which *any* number can be raised to produce 0. As numbers approach 0, their logarithms approach negative infinity.

To find the number corresponding to a given logarithm, called finding the **antilogarithm** (“antilog”); enter the table with the mantissa of the given logarithm and determine the corresponding number, interpolating if necessary. Locate the position of the decimal point by means of the characteristic of the logarithm, in accordance with the rules given above.

110. Multiplication by Logarithms

To *multiply* one number by another, *add* their logarithms and find the antilogarithm of the sum. Thus, to multiply 1,635.8 by 0.0362 by logarithms:

log 1635.8= 3.21373
log 0.0362= 8.55871-10 (add)
log 59.216=11.77244-10 or 1.77244

Thus, 1,635.8×0.0362=59.216. In navigation it is customary to use a slightly modified form, and to omit the -10 where there is no reasonable possibility of confusion, as follows:

1635.8	log	3.21373
0.0362	log	8.55871
59.216	log	1.77244

To *raise a number to a power*, multiply the logarithm of that number by the power indicated, and find the antilogarithm of the product. Thus, to find 13.156^3 by logarithms, using the navigational form:

13.156	log	1.11913	
	x	3	(multiply)
2277.2	log	3.35739	

111. Division by Logarithms

To *divide* one number by another, subtract the logarithm of the divisor from that of the dividend, and find the antilogarithm of the remainder. Thus, to find 0.4637÷28.03 by logarithms, using the navigational form:

0.4637	log	9.66624	
28.03	log (-)	1.44762	(subtract)
0.016543	log	8.21862	

It is sometimes necessary to modify the first logarithm before the subtraction can be made. This would occur in the example given above, for instance, if the divisor and dividend were reversed, so that the problem became 28.03÷0.4637. In this case 10-10 would be added to the logarithm of the dividend, becoming 11.44762-10:

28.03	log	11.44762 - 10
0.4637	log (-)	9.66624 - 10
60.448	log	1.78138

One experienced in the use of logarithms usually carries this change mentally, without showing it in his or her work form:

28.03	log	1.44762
0.4637	log (-)	9.66624
60.448	log	1.78138

Any number can be added to the characteristic as long as that same number is also subtracted. Conversely, any number can be subtracted from the characteristic as long as that same number is also added.

To *extract a root* of a number, divide the logarithm of that number by the root indicated, and find the antilogarithm of the quotient. Thus, to find $\sqrt{7}$ by logarithms:

7	log	0.84510	(÷2)
2.6458	log	0.42255	

To divide a negative logarithm by the root indicated, first modify the logarithm so that the quotient will have a -10.

Thus, to find $\sqrt[3]{0.7}$ by logarithms:

$$\begin{array}{r} 7 \log \underline{29.84510} - 30 \; (\div 3) \\ 0.88792 \log 09.94837 - 10 \end{array}$$

or, carrying the -30 and -10 mentally,

$$\begin{array}{r} 0.7 \log \underline{29.84510} \; (\div 3) \\ 0.88792 \log 9.94837 \end{array}$$

112. Cologarithms

The **cologarithm** ("colog") of a number is the value obtained by subtracting the logarithm of that number from zero, usually in the form 10-10. Thus, the logarithm of 18.615 is 1.26987. The cologarithm is:

$$\begin{array}{r} 10.00000-10 \\ (-)1.26987 \\ 8.73013 - 10 \end{array}$$

Similarly, the logarithm of 0.0018615 is 7.26987 -10, and its cologarithm is:

$$\begin{array}{r} 10.00000 - 10 \\ (-)\,\underline{7.26987 - 10} \\ 2.73013 \end{array}$$

The *cologarithm* of a number is the *logarithm* of the reciprocal of that number. Thus, the cologarithm of 2 is the logarithm of ½. Since division by a number is the same as multiplication by its reciprocal, the use of cologarithms permits division problems to be converted to problems of multiplication, eliminating the need for subtraction of logarithms. This is particularly useful when both multiplication and division are involved in the same problem. Thus, to find $\frac{92.732 \quad 0.0137 \quad 724.3}{0.516 \quad 3941.1}$ by logarithm, one might *add* the logarithms of the three numbers in the numerator, and *subtract* the logarithms of the two numbers in the denominator. If cologarithms are used for the numbers in the denominator, all logarithmic values are added. Thus, the solution might be made as follows:

92.732		log 1.96723
0.0137		log 8.13672
724.3		log 2.85992
0.516	log 9.71265	colog 0.28735
3941.1	log 3.59562	colog 6.40438
0.45248		log 9.65560

113. Various Kinds of Logarithms

As indicated above, **common logarithms** use 10 as the base. These are also called **Brigg's logarithms**. For some purposes, it is convenient to use 2.7182818 approximately (designated e) as the base for logarithms. These are called **natural logarithms** or **Naperian logarithms** ($\log_e$). Common logarithms are shown as $\log_{10}$ when the base might otherwise be in doubt.

Addition and subtraction logarithms are logarithms of the sum and difference of two numbers. They are used when the logarithms of two numbers to be added or subtracted are known, making it unnecessary to find the numbers themselves.

114. Slide Rule

A **slide rule** is a mechanical analog computer. The slide rule is used primarily for multiplication and division, and also for functions such as roots, logarithms and trigonometry. The device is now obsolete with the advent of the hand held electronic calculator in the mid-1970's. Figure 114 depicts a typical slide rule.

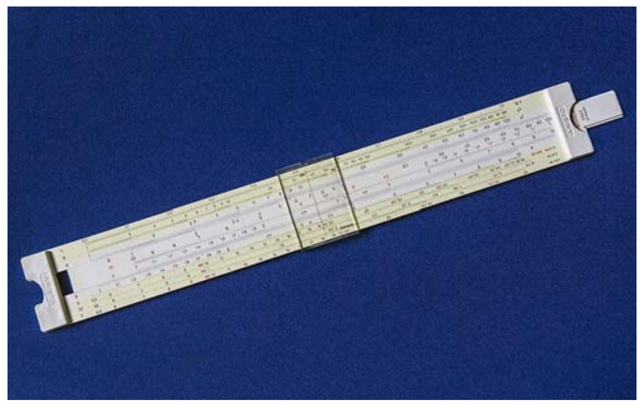


Figure 114. Slide rule. By Jan1959 (own work) via Wikimedia Commons

Slide rules come in many types and sizes, some designed for specific purposes. The most common form consists of an outer "body" or "frame" with grooves to permit a "slide" to be moved back and forth between the two outer parts, so that any graduation of a scale on the slide can be brought opposite any graduation of a scale on the body. A cursor called an "indicator" or "runner" is provided to assist in aligning the desired graduations. In a **circular slide rule** the "slide" is an inner disk surrounded by a larger one, both pivoted at their common center. The scales of a slide rule are *logarithmic*. That is, they increase proportionally to the logarithms of the numbers indicated, rather than to the numbers themselves. This permits addition and subtraction of logarithms by simply measuring off part of the length of the slide from a graduated point on the body, or vice versa. Two or three complete scales within the length of the rule may be provided for finding squares, cubes, square roots, and cube roots.

Properly used, a slide rule can provide quick answers to many of the problems of navigation. However, its precision is usually limited to from two to four significant digits, and should not be used if greater precision is desired.

Great care should be used in placing the decimal point in an answer obtained by slide rule, as the correct location often is not immediately apparent. Its position is usually determined by making a very rough mental solution. Thus, 2.93×8.3 is *about* 3×8=24. Hence, when the answer by slide rule is determined to be "243," it is known that the correct value is 24.3, not 2.43 or 243.

115. Mental Arithmetic

Many of the problems of the navigator can be solved mentally. The following are a few examples.

If the speed is a number divisible into 60 a whole number of times, distance problems can be solved by a simple relationship. Thus, at 10 knots a ship steams 1 mile in $\frac{60}{10} = 6$ minutes. At 12 knots it requires 5 minutes, at 15 knots 4 minutes, etc. As an example of the use of such a relationship, a vessel steaming at 12 knots travels 5.6 miles in 28 minutes, since $\frac{28}{5}=5+\frac{3}{5}=5.6$, or 0.1 mile every half minute.

For relatively short distances, one nautical mile can be considered equal to 6,000 feet. Since one hour has 60 minutes, the speed in hundreds of feet per minute is equal to the speed in knots. Thus, a vessel steaming at 15 knots is moving at the rate of 1,500 feet per minute.

With respect to time, 6 minutes=0.1 hour, and 3 minutes = 0.05 hour. Hence, a ship steaming at 13 knots travels 3.9 miles in 18 minutes (13×0.3), and 5.8 miles in 27 minutes (13×0.45).

In arc units, 6' = 0.1° and 6" = 0.1'. This relationship is useful in rounding off values given in arc units. Thus, 17°23'44"=17°23.7' to the nearest 0.1', and 17.4° to the nearest 0.1°. A thorough knowledge of the six multiplication table is valuable. The 15 multiplication table is also useful, since $15°=1^h$. Hence, 16^h=16×15=240°. This is particularly helpful in quick determination of zone description. Pencil and paper or a table should not be needed, for instance, to decide that a ship at sea in longitude 157°18.4' W is in the (+)10 zone.

It is also helpful to remember that $1°=4^m$ and $1'=4^s$. In converting the LMT of sunset to ZT, for instance, a quick mental solution can be made without reference to a table. Since this correction is usually desired only to the nearest whole minute, it is necessary only to multiply the longitude difference in degrees (to the nearest quarter degree) by four.

VECTORS

116. Scalars and Vector Quantities

A **scalar** is a quantity which has *magnitude* only; a **vector quantity** has both *magnitude* and *direction*. If a vessel is said to have a tank of 5,000 gallons capacity, the number 5,000 is a scalar. As used in this book, *speed* alone is considered a scalar, while *speed* and *direction* are considered to constitute *velocity*, a vector quantity. Thus, if a vessel is said to be steaming at 18 knots, without regard to direction, the number 18 is considered a scalar; but if the vessel is said to be steaming at 18 knots on course 157°, the combination of 18 knots, and 157° constitutes a vector quantity. *Distance* and *direction* also constitute a vector quantity.

A *scalar* can be represented fully by a number. A *vector quantity* vector requires, in addition, an indication of direction. This is conveniently done graphically by means of a straight line, the length of which indicates the *magnitude*, and the direction of which indicates the *direction* of application of the magnitude. Such a line is called a **vector**. Since a straight line has two directions, reciprocals of each other, an arrowhead is placed along or at one end of a vector to indicate the direction represented, unless this is apparent or indicated in some other manner.

117. Addition and Subtraction of Vectors

Two vectors can be *added* by *starting* the second at the termination (rather than the origin) of the first. A common navigational use of vectors is the dead reckoning plot of a vessel. Refer to Figure 117 depicting the addition and subtraction of vectors. If a ship starts at *A* and steams 18 miles on course 090° and then 12 miles on course 060°, it arrives by dead reckoning at *C*. The line *AB* is the vector for the first run, and *BC* is the vector for the second. Point *C* is the position found by *adding* vectors *AB* and *BC*. The vector *AC*, in this case the *course and distance made good*, is the **resultant**. Its value, both in direction and amount, can be determined by measurement. Lines *AB*, *BC*, and *AC* are all **distance vectors**. **Velocity vectors** are used when determining the effect of, or allowing for, current, interconverting true and apparent wind, and solving relative motion problems.

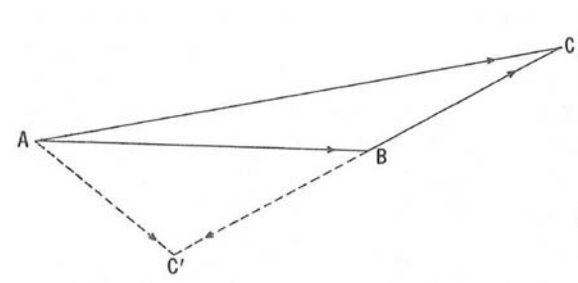


Figure 117. Addition and subtraction of vectors.

The **reciprocal** of a vector has the same magnitude but opposite direction of the vector. To *subtract* a vector, *add* it's reciprocal. This is indicated by the broken lines in Figure 117, in which the vector *BC*' is drawn in the opposite direction to *BC*. In this case the resultant is *AC*'. Subtraction of vectors is involved in some current and wind problems.

ALGEBRA

118. Definitions

Algebra is that branch of mathematics dealing with computation by letters and symbols. It permits the mathematical statement of certain relationships between variables. When numbers are substituted for the letters, algebra becomes arithmetic. Thus if $a=2b$, any value may be assigned to b, and a can be found by multiplying the assigned value by 2. Any statement of equality (as $a=2b$) is an **equation**. Any combination of numbers, letters, and symbols (as $2b$) is a **mathematical expression**.

119. Symbols

As in arithmetic, plus (+) and minus (-) signs are used, and with the same meaning. Multiplication (×) and division (÷) signs are seldom used. In algebra, $a \times b$ is usually written ab, or sometimes a·b. For division $a \div b$ is usually written $\frac{a}{b}$ or a/b. The symbol > means "greater than" and < means "less than." Thus, $a > b$ means "a is greater than b," and $a \geq b$ means "a is equal to or greater than b."

The order of performing the operations indicated in an equation should be observed carefully. Consider the equation $a=b+cd-e/f$. If the equation is to be solved for a, the value cd should be determined by multiplication and e/f by division *before* the addition and subtraction, as each of these is to be considered a single quantity in making the addition and subtraction. Thus, if $cd=g$ and $e/f=h$, the formula can be written $a=b+g-h$.

If an equation including both multiplication and division between plus or minus signs is not carefully written, some doubt may arise as to which process to perform first. Thus, $a \div b \times c$ or $a/b \times c$ may be interpreted to mean either that a/b is to be multiplied by c or that a is to be divided by $b \times c$. Such an equation is better written ac/b if the first meaning is intended, or a/bc if the second meaning is intended.

Parentheses, ( ), may be used for the same purpose or to indicate any group of quantities that is to be considered a single quantity. Thus, a(b+c) is an indication that the sum of b and c is to be multiplied by a. Similarly, a+(b-c)2 indicates that c is first to be subtracted from b, and then the result is to be squared and the value thus obtained added to a. When an expression within parentheses is part of a larger expression which should also be in parentheses, brackets, [ ], are used in place of the outer parentheses. If yet another set is needed, braces, { }, are used.

A quantity written $\sqrt{3}\,ab$ is better written $ab\,\sqrt{3}$ to remove any suggestion that the square root of $3ab$ is to be found.

120. Addition and Subtraction

Addition and subtraction.-A plus sign before an expression in parentheses means that each term retains its sign as given. Thus, $a + (b + c - d)$ is the same as $a + b + c - d$. A minus sign preceding the parentheses means that each sign within the parentheses is to be reversed. For example, $a - (b + c - d) = a - b - c + d$.

In any equation involving addition and subtraction, similar terms can be combined. Thus, $a+b+c+b-2c-d=a+2b-c-d$. Also, $a+3ab+a^2-b-ab=a+2ab+a^2-b$. That is, to be combined, the terms must be truly alike, for a cannot be combined with ab, or with a^2.

Equal quantities can be added to or subtracted from both members of an equation without disturbing the equality. Thus, if $a=b$, $a+2=b+2$, or $a+x=b+x$. If $x=y$, then $a+x=b+y$.

121. Multiplication and Division

When an expression in parentheses is to be multiplied by a quantity outside the parentheses, each quantity separated by a plus or minus sign within the parentheses should be multiplied separately. Thus, $a(b+cd-e/f)$ may be written $ab+acd-ae/f$. Any quantity appearing in *every* term of one member of an equation can be separated out by **factoring**, or dividing each term by the common quantity. Thus, if $a = bc + \frac{bd}{e} - b^2 + b$, the equation may be written $a = b\left(c + \frac{d}{e} - b + 1\right)$.

Note that $\frac{b}{b}$ = and $\frac{b^2}{b} = b$. This is the inverse of multiplication: $a \times 1 = a$, but $a \times a = a^2$. Also, $a^2 \times a^3 = a^5$; and $\frac{a^7}{a^2} = a^5$. Thus, in multiplying a power of a number by a power of the same number, the powers are added, or, stated mathematically, $a^m \text{ x } a^n = a^{m+n}$. In division, $\frac{a^m}{a^n} = a^{m-n}$, or the exponents are subtracted. If n is greater than m, a *negative* exponent results. A value with a negative exponent is equal to the reciprocal of the same value with a positive exponent. Thus, $a^{-n} = \frac{1}{a^n}$ and $\frac{a^2 b^{-3}}{c} = \frac{a^2}{b^3 c}$.

In raising to a power a number with an exponent, the two exponents are multiplied. Thus, $(a^2)^3 = a^{2\text{x}3} = a^6$, or $(a^n)^m = a^{nm}$. The inverse is true in extracting a root. Thus, $\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{0.667}$, or $\sqrt[m]{a^n} = a^{\frac{n}{m}}$.

Both members of an equation can be multiplied or divided by equal quantities without disturbing the equality, excluding division by zero or some expression equal to zero. Thus, if a=b+c, 2a=2(b+c), or if x=y, ax=y(b+c) and $\frac{a}{x} = \frac{b+c}{y}$. Sometimes there is more than one answer to an equation. Division by one of the unknowns may eliminate one of the answers.

Both members of an equation can be raised to the same power, and like roots of both members can be taken, without disturbing the equality. Thus, if a=b+c, a^2=(b+c)^2, or if x=y, a^x=(b+c)^y. This is *not* the same as a^x=b^y+c^y. Similarly, if a=b+c, $\sqrt{a} = \sqrt{b+c}$, or if x=y, $\sqrt[x]{a} = \sqrt[y]{b+c}$. Again, $\sqrt[y]{b+c}$ is *not* equal to $\sqrt[y]{b} + \sqrt[y]{c}$, as a numerical example will indicate: $\sqrt{100} = \sqrt{64+36}$, but $\sqrt{100}$ does not equal $\sqrt{64} + \sqrt{36}$.

If two quantities to be multiplied or divided are both positive or both negative, the result is positive. Thus, (+a)x(+b)=ab and $\frac{-a}{-b}$=+$\frac{a}{b}$. But if, the signs are opposite, the answer is negative. Thus, (+a)x(-b)=-ab, and $\frac{-a}{+\mathrm{b}}$=-$\frac{a}{b}$; also, (-a)x(+b)=-ab, and $\frac{+\mathrm{a}}{-b}$=-$\frac{a}{b}$.

In expressions containing both parentheses and brackets, or both of these and braces, the innermost symbols are removed first. Thus, $-\left\{6z - \frac{x(\mathrm{x}+4) - 5y}{y}\right\}$=-$\left\{6z - \frac{[x^2 + 4x - 5y]}{y}\right\}$=-$\left\{6z - \frac{x^2}{y} - \frac{4x}{y} + 5\right\}$=-6z + $\frac{x^2}{y} + \frac{4x}{y} - 5$.

122. Fractions

To add or subtract two or more fractions, convert each to an expression having the same denominator, and then add the numerators.

Thus, $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf}$. That is, both numerator and denominator of each fraction are multiplied by the denominator of the other remaining fractions.

To multiply two or more fractions, multiply the numerators by each other, and also multiply the denominators by each other. Thus, $\frac{a}{b} \quad \frac{c}{d} \quad \frac{e}{f} = \frac{ace}{bdf}$.

To divide two fractions, invert the divisor and multiply. Thus, $\frac{a}{b} \quad \frac{c}{d} = \frac{a}{b} \quad \frac{d}{c} = \frac{ad}{bc}$.

If the same factor appears in all terms of a fraction, it can be factored out without changing the value of the fraction. Thus, $\frac{ab + ac + ad}{ae - af} = \frac{b + c + d}{e - f}$. This is the same as factoring a from the numerator and denominator separately. That is, $\frac{ab + ac + ad}{ae - af} = \frac{a(\mathrm{b+c+d})}{a(\mathrm{e-\ f})}$, but since $\frac{a}{a}$=1, this part can be removed, and the fraction appears as above.

123. Transposition

It is sometimes desirable to move terms of an expression from one side of the equals sign (=) to the other. This is called **transposition**, and to move one term is to **transpose** it. If the term to be moved is preceded by a plus or a minus sign, this sign is reversed when the term is transposed. Thus, if $a = b + c$, then $a - b = c$, $a - c = b$, $-b = c - a$, $-b - c = -a$, $-b - c = -a$, etc. Note that the signs of all terms can be reversed without destroying the equality, for if $a = b$, $b = a$. Thus, if all terms to the left of the equals sign are exchanged for all those to the right, no change in sign need take place, yet if each is moved individually, the signs reverse. For instance, if $a = b + c$, $-b - c = -a$. If each term is multiplied by -1, this becomes $b + c = a$.

A term which is to be multiplied or divided by *all* other terms on its side of the equation can be transposed if it is also moved from the numerator to the denominator, or vice versa. Thus, if $a = \frac{b}{c}$, then $ac = b$, $c = \frac{b}{a}$, $\frac{1}{b} = \frac{1}{ac}$, $\frac{c}{b} = \frac{1}{a}$, etc. (Note that $a = \frac{a}{1}$.) The same result could be obtained by multiplying both sides of an equation by the same quantity. For instance, if both sides of $a = \frac{b}{c}$ are multiplied by c, the equation becomes $ac = \frac{bc}{c}$ and since any number (except zero) divided by itself is unity, $\frac{c}{c} = 1$, and the equation becomes $ac = b$, as given above. Note, also, that *both* sides of an equation can be *inverted* without destroying the relationship, for if $a = b$, $\frac{a}{1} = \frac{b}{1}$, and $\frac{1}{b} = \frac{1}{a}$ or $\frac{1}{a} = \frac{1}{b}$. This is accomplished by transposing *all* terms of an equation.

Note that in the case of transposition by changing the plus or minus sign, an entire expression must be changed, and not a part of it. Thus, if $a = bc + d$, $a - bc = d$, but it is not true that $a + b = c + d$. Similarly, a term to be transposed by reversing its multiplication-division relationship must bear that relationship to *all* other terms on its side of the equation. That is, if $a = bc + d$, it is *not* true that $\frac{a}{b} = c + d$, or that $\frac{a}{bc} = d$, but $\frac{a}{bc + d} = 1$, if $a = b(cd + e)$.

124. Ratio and Proportions

If the relationship of a to b is the same as that of c to d, this fact can be written $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$. Either side of this equation, $\frac{a}{b}$ or $\frac{c}{d}$ is called a **ratio** and the whole equation is called a **proportion**. When a ratio is given a numerical value, it is often expressed as a decimal or as a percentage. Thus, if $\frac{a}{b} = \frac{1}{4}$ (that is, $a = 1$, $b = 4$), the ratio might be expressed as 0.25 or as 25 percent.

Since a ratio is a fraction, it can be handled as any other fraction.

GEOMETRY

125. Definition

Geometry deals with the properties, relations, and measurement of lines, surfaces, solids, and angles. **Plane geometry** deals with plane figures, and **solid geometry** deals with three–dimensional figures.

A **point**, considered mathematically, is a place having position but no extent. It has no length, breadth, or thickness. A point in motion produces a **line**, which has length, but neither breadth nor thickness. A **straight or right line** is the shortest distance between two points in space. A line in motion in any direction except along itself produces a **surface**, which has length and breadth, but not thickness. A **plane surface** or **plane** is a surface without curvature. A straight line connecting any two of its points lies wholly within the plane. A plane surface in motion in any direction except within its plane produces a **solid**, which has length, breadth, and thickness. **Parallel lines** or surfaces are those which are everywhere equidistant. **Perpendicular lines** or surfaces are those which meet at right or 90° angles. A perpendicular may be called a **normal**, particularly when it is perpendicular to the tangent to a curved line or surface at the point of tangency. All points equidistant from the ends of a straight line are on the perpendicular bisector of that line. The shortest distance from a point to a line is the length of the perpendicular between them.

126. Angles

An **angle** is formed by two straight lines which meet at a point. It is measured by the arc of a circle intercepted between the two lines forming the angle, the center of the circle being at the point of intersection. In Figure 126a, the angle formed by lines *AB* and *BC*, may be designated "angle *B*," "angle *ABC*," or "angle *CBA*"; or by Greek letter as "angle α." The three letter designation is preferred if there is more than one angle at the point. When three letters are used, the middle one should always be that at the **vertex** of the angle.

An **acute angle** is one less than a right angle (90°).

A **right angle** is one whose sides are perpendicular (90°).

An **obtuse angle** is one greater than a right angle (90°) but less than 180°.

A **straight angle** is one whose sides form a continuous straight line (180°).

A **reflex angle** is one greater than a straight angle (180°) but less than a circle (360°). Any two lines meeting at a point form two angles, one less than a straight angle of 180° (unless exactly a straight angle) and the other greater than a straight angle.

An **oblique angle** is any angle not a multiple of 90°.

Two angles whose sum is a right angle (90°) are **complementary angles**, and either is the **complement** of the other.

Two angles whose sum is a straight angle (180°) are **supplementary angles**, and either is the **supplement** of the other.

Two angles whose sum is a circle (360°) are **explementary angles**, and either is the **explement** of the other. The two angles formed when any two lines terminate at a common point are explementary.

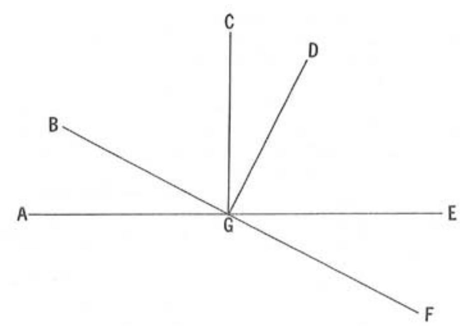


Figure 126a. Acute, right, and obtuse angles.

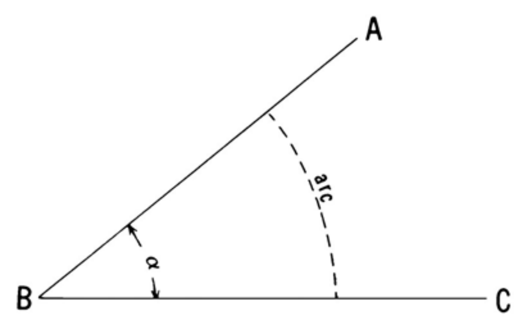


Figure 126b. An angle.

If the sides of one angle are perpendicular to those of another, the two angles are either equal or supplementary. Also, if the sides of one angle are parallel to those

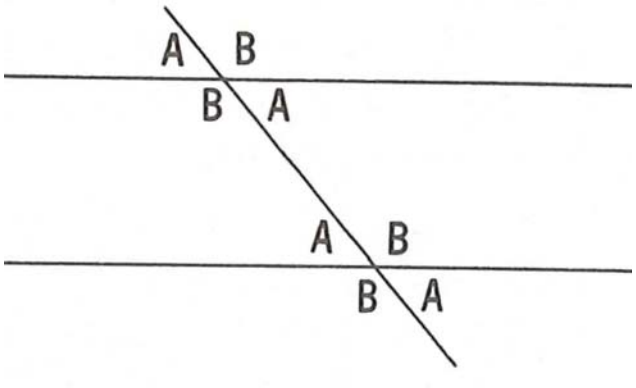


Figure 126c. Angles formed by a transversal.

of another, the two angles are either equal or supplementary.

When two straight lines intersect, forming four angles, the two opposite angles, called **vertical angles**, are equal. Angles which have the same vertex and lie on opposite sides of a common side are **adjacent angles**. Adjacent angles formed by intersecting lines are supplementary, since each pair of adjacent angles forms a straight angle. Thus, in Figure 126a, lines *AE* and BF intersect at *G*. Angles *AGB* and *EGF* form a pair of equal acute vertical angles, and *BGE* and *AGF* form a pair of equal obtuse vertical angles.

A **transversal** is a line that intersects two or more other lines. If two or more parallel lines are cut by a transversal, groups of adjacent and vertical angles are formed, as shown in Figure 126c. In this situation, all acute angles (*A*) are equal, all obtuse angles (*B*) are equal, and each acute angle is supplementary to each obtuse angle.

A **dihedral angle** is the angle between two intersecting planes.

127. Triangles

A **plane triangle** is a closed figure formed by three straight lines, called **sides**, which meet at three points called **vertices**. The vertices are labeled with capital letters and the sides with lowercase letters, as shown in Figure 127a, which depicts a triangle.

An **equilateral triangle** is one with its three sides equal in length. It must also be **equiangular**, with its three angles equal.

An **isosceles triangle** is one with two equal sides, called **legs**. The angles opposite the legs are equal. A line which **bisects** (divides into two equal parts) the unequal angle of an isosceles triangle is the perpendicular bisector of the opposite side, and divides the triangle into two equal right triangles.

A **scalene triangle** is one with no two sides equal. In such a triangle, no two angles are equal.

An **acute triangle** is one with three acute angles.

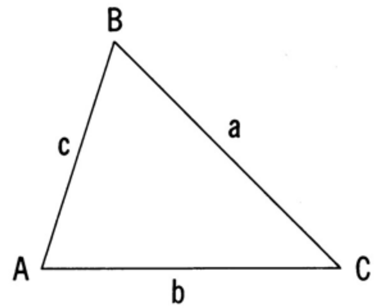


Figure 127a. A triangle.

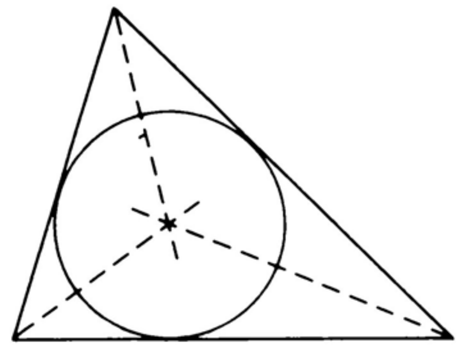


Figure 127b. A circle inscribed in a triangle.

A **right triangle** is one having a right angle. The side opposite the right angle is called the **hypotenuse**. The other two sides may be called **legs**. A plane triangle can have only one right angle.

An **obtuse triangle** is one with an obtuse angle. A plane triangle can have only one obtuse angle.

An **oblique triangle** is one which does not contain a right angle.

The **altitude** of a triangle is a line or the distance from any vertex perpendicular to the opposite side.

A **median** of a triangle is a line from any vertex to the center of the opposite side. The three medians of a triangle meet at a point called the **centroid** of the triangle. This point divides each median into two parts, that part between the centroid and the vertex being twice as long as the other part.

Lines bisecting the three *angles* of a triangle meet at a point which is equidistant from the three sides, which is the center of the **inscribed circle**, as shown in Figure 127b. This point is of particular interest to navigators because it is the point theoretically taken as the fix when three lines of position of equal weight and having only random errors do not meet at a common point. In practical navigation, the point is found visually, not by construction, and other factors often influence the chosen fix position.

The perpendicular bisectors of the three *sides* of a triangle meet at a point which is equidistant from the three vertices, which is the center of the **circumscribed circle**, the circle through the three vertices and the smallest circle

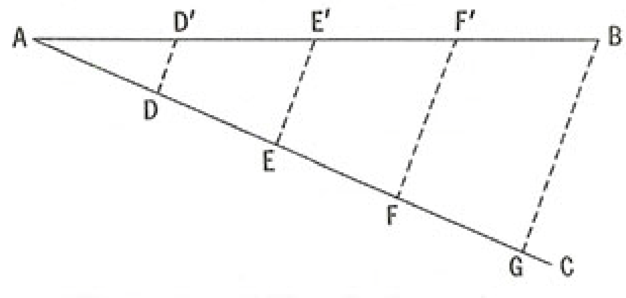


Figure 127c. Dividing a line into equal parts.

which can be drawn enclosing the triangle. The center of a circumscribed circle is *within* an acute triangle, *on the hypotenuse* of a right triangle, and *outside* an obtuse triangle.

A line connecting the mid–points of two sides of a triangle is always parallel to the third side and half as long. Also, a line parallel to one side of a triangle and intersecting the other two sides divides these sides proportionally. This principle can be used to divide a line into any number of equal or proportional parts. Refer to Figure 127c, which depicts dividing a line into equal parts. Suppose it is desired to divide line *AB* into four equal parts. From *A* draw any line *AC*. Along *C* measure four equal parts of any convenient lengths (*AD*, *DE*, *EF*, and *FG*). Draw *GB*, and through *F*, *E*, and *D* draw lines parallel to *GB* and intersecting *AB*. Then *AD'*, *D' E'*, *E' F'*, and *F' B* are equal and *AB* is divided into four equal parts.

The sum of the angles of a plane triangle is always 180°. Therefore, the sum of the acute angles of a right triangle is 90°, and the angles are complementary. If one side of a triangle is extended, the **exterior angle** thus formed is supplementary to the adjacent **interior angle** and is therefore equal to the sum of the two non adjacent angles. If two angles of one triangle are equal to two angles of another triangle, the third angles are also equal, and the triangles are **similar**. If the area of one triangle is equal to the area of another, the triangles are **equal**. Triangles having equal bases and altitudes also have equal areas. Two figures are **congruent** if one can be placed over the other to make an exact fit. Congruent figures are both similar and equal. If any side of one triangle is equal to any side of a similar triangle, the triangles are congruent. For example, if two right triangles have equal sides, they are congruent; if two right triangles have two corresponding sides equal, they are congruent. Triangles are congruent only if the sides and angles are equal.

The sum of two sides of a plane triangle is always greater than the third side; their difference is always less than the third side.

The area of a triangle is equal to 1/2 of the area of the polygon formed from its base and height. If A = area, b = one of the legs of a right triangle or the base of any plane triangle, h = altitude, c = the hypotenuse of a right triangle, a = the other leg of a right triangle, and S = the sum of the interior angles:

$$\text{Area of plane triangle } A = \frac{bh}{2}$$

Sum of interior angles of plane triangle: S = 180°

The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides, or $a^2 + b^2 = c^2$. Therefore the length of the hypotenuse of plane right triangle can be found by the formula:

$$c = \sqrt{a^2 + b^2}$$

128. Polygons

A **polygon** is a closed plane figure made up of three or more straight lines called **sides**. A polygon with three sides is a **triangle**, one with four sides is a **quadrilateral**, one with five sides is a **pentagon**, one with six sides is a **hexagon**, and one with eight sides is an **octagon**. An **equilateral polygon** has equal sides. An **equiangular polygon** has equal interior angles. A **regular polygon** is both equilateral and equiangular. As the number of sides of a regular polygon increases, the figure approaches a circle.

A **trapezoid** is a quadrilateral with one pair of opposite sides parallel and the other pair not parallel. A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel. Any side of a parallelogram, or either of the parallel sides of a trapezoid, is the **base** of the figure. The perpendicular distance from the base to the opposite side is the altitude. A **rectangle** is a parallelogram with four right angles. (If anyone is a right angle, the other three must be, also.) A **square** is a rectangle with equal sides. A **rhomboid** is a parallelogram with oblique angles. A **rhombus** is a rhomboid with equal sides.

The sum of the exterior angles of a convex polygon (one having no interior reflex angles), made by extending each side in one direction only (consistently), is 360°.

A **diagonal** of a polygon is a straight line connecting any two vertices which are not adjacent. The diagonals of a parallelogram bisect each other.

The **perimeter** of a polygon is the sum of the lengths of its sides.

If A = area, s = the side of a square, a = that side of a rectangle adjacent to the base or that side of a trapezoid parallel to the base, b = the base of a quadrilateral, h = the altitude of a parallelogram or trapezoid, S = the sum of the angles of a polygon, and n = the number of sides of a polygon:

Area of a square: $A = s^2$

Area if a rectangle: $A = ab$

Area of a parallelogram: $A = bh$

Area of a trapezoid: $A = \frac{(a+b)h}{2}$

Sum of angles in convex polygon: $S = (n-2)180°$.

129. Circles

A **circle** is a plane, closed curve, all points of which are equidistant from a point within, called the **center**. See Figure 129 depicting elements of a circle.

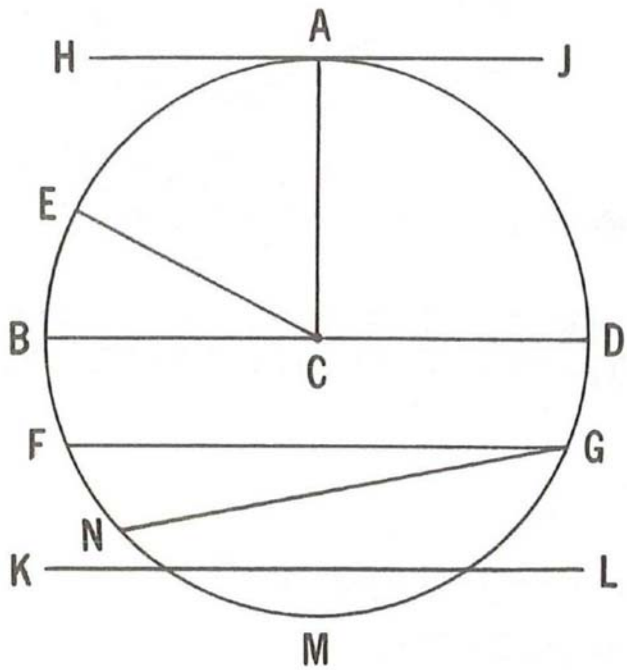


Figure 129. Elements of a circle.

The distance around a circle is called the **circumference**. Technically the length of this line is the **perimeter**, although the term "circumference" is often used. An arc is part of a circumference. A **major arc** is more than a semicircle (180°), a **minor arc** is less than a semicircle (180°). A **semi–circle** is half a circle (180°), a **quadrant** is a quarter of a circle (90°), a **quintant** is a fifth of a circle (72°), a **sextant** is a sixth of a circle (60°), an **octant** is an eighth of a circle (45°). Some of these names have been applied to instruments used by navigators for measuring altitudes of celestial bodies because of the part of a circle used for the length of the arc of the instrument.

Concentric circles have a common center. A **radius** (plural **radii**) or **semidiameter** is a straight line connecting the center of a circle with any point on its circumference. In Figure 129, *CA*, *CB*, *CD*, and *CE* are radii

A **diameter** of a circle is a straight line passing through its center and terminating at opposite sides of the circumference, or two radii in opposite directions (*BCD*, Figure 129). It divides a circle into two equal parts. The ratio of the length of the circumference of any circle to the length of its diameter is 3.14159+, or π (the Greek letter pi), a relationship that has many useful applications.

A **sector** is that part of a circle bounded by two radii and an arc. In Figure 129, *BCE*, *ECA*, *ACD*, *BCA*, and *ECD* are sectors.The angle formed by two radii is called a **central angle**. Any pair of radii divides a circle into sectors, one less than a semicircle (180°) and the other greater than a semicircle (unless the two radii form a diameter).

A **chord** is a straight line connecting any two points on the circumference of a circle (*FG*, *GN* in Figure 129). Chords equidistant from the center of a circle are equal in length.

A **segment** is the part of a circle bounded by a chord and the intercepted arc (*FGMF*, *NGMN* in Figure 129). A chord divides a circle into two segments, one less than a semicircle (180°), and the other greater than a semicircle (unless the chord is a diameter). A diameter perpendicular to a chord bisects it, its arc, and its segments. Either pair of vertical angles formed by intersecting chords has a combined number of degrees equal to the sum of the number of degrees in the two arcs intercepted by the two angles.

An **inscribed angle** is one whose vertex is on the circumference of a circle and whose sides are chords (*FGN* in Figure 129). It has half as many degrees as the arc it intercepts. Hence, an angle inscribed in a semicircle is a right angle if its sides terminate at the ends of the diameter forming the semicircle.

A **secant** of a circle is a line intersecting the circle, or a chord extended beyond the circumference (*KL* in Figure 129).

A **tangent** to a circle is a straight line, in the plane of the circle, which has only one point in common with the circumference (*HJ* in Figure 129). A tangent is perpendicular to the radius at the **point of tangency** (A in Figure 129). Two tangents from a common point to opposite sides of a circle are equal in length, and a line from the point to the center of the circle bisects the angle formed by the two tangents. An angle formed outside a circle by the intersection of two tangents, a tangent and a secant, or two secants has *half* as many degrees as the *difference* between the two intercepted arcs. An angle formed by a tangent and a chord, with the apex at the point of tangency, has half as many degrees as the arc it intercepts. A **common tangent** is one tangent to more than one circle. Two circles are tangent to each other if they touch at one point only. If of different sizes, the smaller circle may be either inside or outside the larger one.

Parallel lines intersecting a circle intercept equal arcs.

If A = area; r = radius; d = diameter; C = circumference; s = linear length of an arc; a = angular length of an arc, or the angle it subtends at the center of a circle, in degrees; b = angular length of an arc, or the angle it subtends at the center of a circle, in radians; *rad* = radians and *sin* = sine:

Circumference of a circle $C = 2\pi r = \pi d = 2\pi$ rad

Area of circle $A = \pi r^2 = \frac{\pi d^2}{4}$

Area of sector $= \frac{\pi r^2 a}{360} = \frac{r^2 b}{2} = \frac{rs}{2}$

Area of segment $= \frac{r^2(b - \sin a)}{2}$

130. Polyhedrons

A **polyhedron** is a solid having plane sides or faces.

A **cube** is a polyhedron having six square sides.

A **prism** is a solid having parallel, similar, equal, plane geometric figures as bases, and parallelograms as sides. By extension, the term is also applied to a similar solid having nonparallel bases, and trapezoids or a combination of trapezoids and parallelograms as sides. The **axis** of a prism is the straight line connecting the centers of its bases. A **right prism** is one having bases perpendicular to the axis. The sides of a right prism are rectangles. A **regular prism** is a right prism having regular polygons as bases. The **altitude** of a prism is the perpendicular distance between the planes of its bases. In the case of a right prism it is measured along the axis.

A **pyramid** is a polyhedron having a polygon as one end, the **base**; and a point, the **apex**, as the other; the two ends being connected by a number of triangular sides or **faces**. The **axis** of a pyramid is the straight line connecting the apex and the center of the base. A **right pyramid** is one having its base perpendicular to its axis. A **regular pyramid** is a right pyramid having a regular polygon as its base. The altitude of a pyramid is the perpendicular distance from its apex to the plane of its base. A **truncated pyramid** is that portion of a pyramid between its base and a plane intersecting all of the faces of the pyramid.

If A = area, s = edge of a cube or slant height of a regular pyramid (from the center of one side of its base to the apex), V = volume, a = side of a polygon, h = altitude, P = perimeter of base, n = number of sides of polygon, B = area of base, and r = perpendicular distance from the center of side of a polygon to the center of the polygon:

Cube:

Area of each face: $A = s^2$

Total area of all faces: $A = 6s^2$

Volume: $V = s^3$

Regular prism:

Area of each face: $A = ah$

Total area of all faces: $A = Ph = nah$

Area of each base: $B = \frac{nar}{2}$

Total area of both bases: $A = nar$

Volume: $V = Bh = \frac{narh}{2}$

Regular pyramid:

Area of each face: $A = \frac{as}{2}$

Total area of all faces: $A = \frac{nas}{2}$

Area of base: $B = \frac{nar}{2}$

Volume: $V = \frac{Bh}{3} = \frac{narh}{6}$

131. Cylinders

A **cylinder** is a solid having two parallel plane **bases** bounded by closed congruent curves, and a surface formed by an infinite number of parallel lines, called **elements**, connecting similar points on the two curves. A cylinder is similar to a prism, but with a curved lateral surface, instead of a number of flat sides connecting the bases. The axis of a cylinder is the straight line connecting the centers of the bases. A **right cylinder** is one having bases perpendicular to the axis. A **circular cylinder** is one having circular bases. The **altitude** of a cylinder is the perpendicular distance between the planes of its bases. The **perimeter** of a base is the length of the curve bounding it.

If A = area, P = perimeter of base, h = altitude, r = radius of a circular base, B = area of base, and V = volume, then for a right circular cylinder:

Lateral area: $A = Ph = 2\pi rh$

Area of each base: $B = \pi r^2$

Total area, both bases: $A = 2\pi r^2$

Volume: $V = Bh = \pi r^2 h$

132. Cones

A **cone** is a solid having a plane **base** bounded by a closed curve, and a surface formed by lines, called **elements**, from every point on the curve to a common point called the **apex**. A cone is similar to a pyramid, but with a curved surface connecting the base and apex, instead of a number of flat sides. The **axis** of a cone is the straight line connecting the apex and the center of the base. A **right cone** is one having its base perpendicular to its axis. A **circular cone** is one having a circular base. The **altitude** of a cone is the perpendicular distance from its apex to the plane of its base. A **frustum** of a cone is that portion of the cone between its base and any parallel plane intersecting all elements of the cone. A **truncated cone** is that portion of a cone between its base and any nonparallel plane which intersects all elements of the cone but does not intersect the base.

If A = area, r = radius of base, s = slant height or length of element, B = area of base, h = altitude, and V = volume, then for a right circular cone:

Lateral area: $A = \pi rs$

Area of base: $B = \pi r^2$

Slant height: $s = \sqrt{r^2 + h^2}$

Volume: $V = \frac{Bh}{3} = \frac{\pi r^2 h}{3}$

133. Conic Sections

If a right circular cone of indefinite extent is intersected by a plane perpendicular to the axis of the cone the line of intersection of the plane and the surface of the cone is a circle. Refer to Figure 133a for a depiction of conic sections.

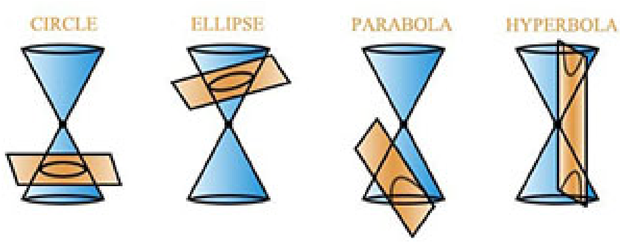


Figure 133a. Conic sections.

If an intersecting plane is tilted to some position, the intersection is an **ellipse** or flattened circle, see Figure 133b. The longest diameter of an ellipse is called its **major axis**, and half of this is its **semimajor axis**, which is identified by the letter "**a**" in Figure 133b. The shortest diameter of an ellipse is called its **minor axis**, and half of this is its semiminor axis, which is identified by the letter "**b**" in figure Figure 133b. Two points, F and F', called **foci** (singular **focus**) or **focal points**, on the major axis are so located that the sum of their distances from any point P on the curve is equal to the length of the major axis. That is $PF + PF' = 2a$ (Figure 133b). The **eccentricity** (**e**) of an ellipse is equal to $\frac{c}{a}$, where c is the distance from the center to one of the foci ($c = CF = CF'$). It is always greater than 0 but less than 1.

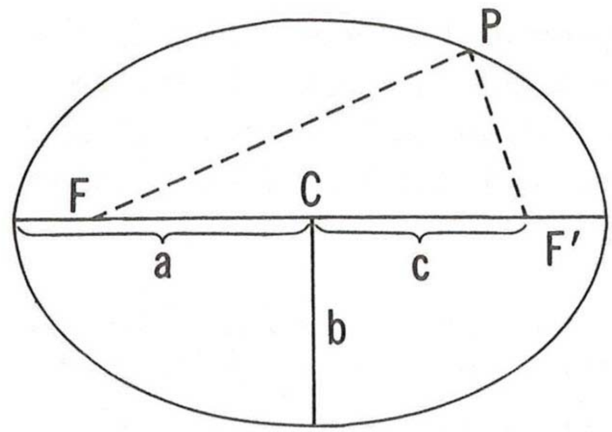


Figure 133b. An ellipse.

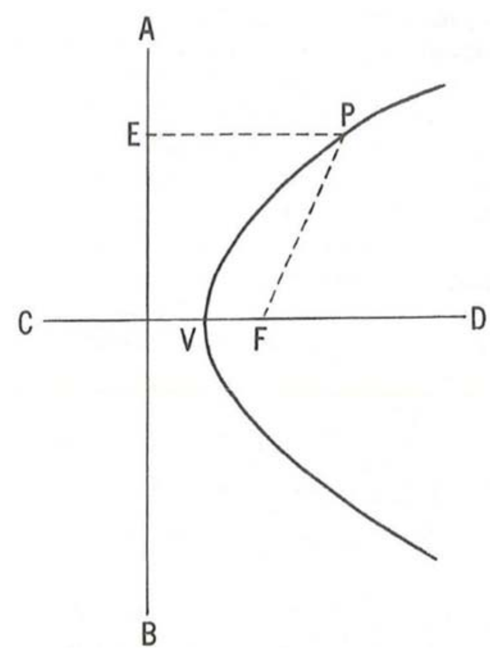


Figure 133c. A parabola.

If an intersecting plane is parallel to one element of the cone the intersection is a **parabola**, see Figure 133c. Any point P on a parabola is equidistant from a fixed point F, called the **focus** or **focal point**, and a fixed straight line, AB, called the **directrix**. Thus, for any point P, $PF = PE$. The point midway between the focus F and the directrix AB is called the **vertex**, V. The straight line through F and V is called the **axis**, CD. This line is perpendicular to the directrix AB. The **eccentricity** (e) of a parabola is 1.

If the elements of the cone are extended to form a second cone having the same axis and apex but extending in the opposite direction, and the intersecting plane is tilted beyond the position forming a parabola, so that it intersects both curves, the intersections of the plane with the cones is a **hyperbola**, see Figure 133d. There are two intersections or branches of a hyperbola, as shown. At any point P on either branch, the difference in the distance from two fixed points called **foci** or **focal points**, F and F', is constant and equal to the shortest distance between the two branches. That is, $PF - PF' = 2a$ (Figure 133d). The straight line through F and F' is called the **axis**. The **eccentricity** (**e**) of a hyperbola is the ratio $\frac{c}{a}$ (Figure 133d). It is always greater than 1.

Each branch of a hyperbola approaches ever closer to, but never reaches, a pair of intersecting straight lines, AB and CD, called **asymptotes**. These intersect at G.

The various conic sections bear an eccentricity relationship to each other. The eccentricity of a circle is 0, that of an ellipse is greater than 0 but less than 1; that of a parabola or straight line (a limiting case of a parabola) is 1, and that of a hyperbola is greater than 1.

If e = eccentricity, A = area, a = semimajor axis of an ellipse or half the shortest distance between the two branches of a hyperbola, b = the semiminor axis of an ellipse, and c = the distance between the center of an ellipse and one of its focal points or the distance between the focal point of a hyperbola and the intersection of its asymptotes:

Circle:

Eccentricity: $e = 0$

Ellipse:

Area: $A = \pi ab$

Eccentricity: $e = \frac{c}{a}$, greater that 0, but less than 1.

Parabola:

Eccentricity: $e = 1$

Hyperbola:

Eccentricity: $e = \frac{c}{a}$, greater than 1.

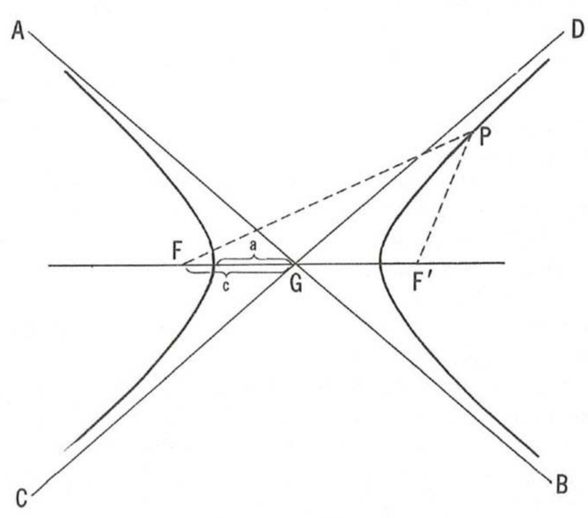


Figure 133d. A hyperbola.

When cones are intersected by some surface other than a plane, as the curved surface of the earth, the resulting sections do not follow the relationships given above, the amount of divergence therefrom depending upon the individual circumstances.

134. Spheres

A **sphere** is a solid bounded by a surface every point of which is equidistant from a point within called the **center**. It may also be formed by rotating a circle about any diameter.

A **radius** or **semidiameter of a sphere** is a straight line connecting its center with any point on its surface. A **diameter** of a sphere is a straight line through its center and terminated at both ends by the surface of the sphere. The poles of a sphere are the ends of a diameter.

The intersection of a plane and the surface of a sphere is a circle, a **great circle** if the plane passes through the center of the sphere, and a **small circle** if it does not. The shorter arc of the great circle between two points on the surface of a sphere is the shortest distance, on the surface of the sphere, between the points. Every great circle of a sphere bisects every other great circle of that sphere. The **poles** of a circle on a sphere are the extremities of the sphere's diameter which is perpendicular to the plane of the circle. All points on the circumference of the circle are equidistant from either of its poles. In the ease of a great circle, *both* poles are 90° from any point on the circumference of the circle. Any great circle may be considered a **primary**, particularly when it serves as the origin of measurement of a coordinate. The great circles through its poles are called **secondary**. Secondaries are perpendicular to their primary.

A **spherical triangle** is the figure formed on the surface of a sphere by the intersection of three great circles. The lengths of the sides of a spherical triangle are measured in degrees, minutes, and seconds, as the angular lengths of the arcs forming them. The sum of the three sides is always less than 360°. The sum of the three angles is always *more* than 180° and *less* than 540°.

A **lune** is the part of the surface of a sphere bounded by halves of two great circles.

A **spheroid** is a flattened sphere, which may be formed by rotating an ellipse about one of its axes. An **oblate spheroid**, such as the earth, is formed when an ellipse is rotated about its minor axis. In this case the diameter along the axis of rotation is less than the major axis. A **prolate spheroid** is formed when an ellipse is rotated about its major axis. In this case the diameter along the axis of rotation is greater than the minor axis.

If A = area, r = radius, d = diameter, and V = volume of a sphere:

Area: $A = 4\pi r^2 = \pi d^2$

Volume: $V = \frac{4\pi r^3}{3} = \frac{\pi d^3}{6}$

If A = area, a = semimajor axis, b = semiminor axis, e = eccentricity, and V = volume of an oblate spheroid:

Area: $A = 4\pi a^2\left(1 - \frac{e^2}{3} - \frac{e^4}{15} - \frac{e^6}{35} - \ldots\right)$

Eccentricity: $e = \sqrt{\frac{a^2 - b^2}{a^2}}$

Volume: $V = \frac{4\pi a^2 b}{3}$

135. Coordinates

Coordinates are magnitudes used to define a position. Many different types of coordinates are used. Important navigational ones are described below.

If a position is known to be at a stated point, no magnitudes are needed to identify the position, although they may be required to locate the point. Thus, if a vessel is at port *A*, its position is known if the location of port *A* is known, but latitude and, longitude may be needed to locate port *A*.

If a position is known to be on a given line, a single magnitude (coordinate) is needed to identify the position if an origin is stated or understood. Thus, if a vessel is known to be *south* of port *B*, it is known to be on a line extending southward from port *B*. If its distance from port *B* is known, and the position of port *B* is known, the position of the vessel is uniquely defined.

If a position is known to be on a given surface, two magnitudes (coordinates) are needed to define the position. Thus, if a vessel is known to be on the surface of the earth, its position can be identified by means of latitude and longitude. Latitude indicates its angular distance north or south of the equator, and longitude its angular distance east or west of the prime meridian.

If nothing is known regarding a position other than that it exists in space, three magnitudes (coordinates) are needed to define its position. Thus, the position of a submarine may be defined by means of latitude, longitude, and depth below the surface.

Each coordinate requires an origin, either stated or implied. If a position is known to be on a given plane, it might be defined by means of its distance from each of two intersecting lines, called **axes**. These are called **rectangular coordinates**. In Figure 135a, *OY* is called the **ordinate**, and *OX* is called the **abscissa**. Point *O* is the **origin**, and lines *OX* and *OY* the axes (called the *X* and *Y* axes, respectively). Point *A* is at position *x*, *y*. If the axes are not perpendicular but the lines *x* and *y* are drawn parallel to the axes, **oblique coordinates** result. Either type are called **Cartesian coordinates**. A three–dimensional system of Cartesian coordinates, with *X*, *Y*, and *Z* axes, is called **space coordinates**.

Another system of plane coordinates in common usage consists of the *direction* and *distance* from the origin (called the **pole**), as shown in Figure 135b. A line extending in the direction indicated is called a **radius vector**. Direction and distance from a fixed point constitute **polar coordinates**, sometimes called the rho– (the Greek ρ, to indicate distance) theta (the Greek θ, to indicate direction) system. An example of its use is the radar scope.

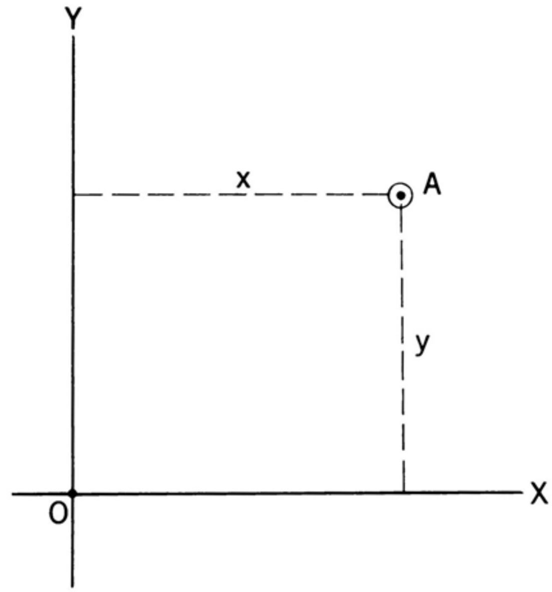


Figure 135a. Rectangular coordinates.

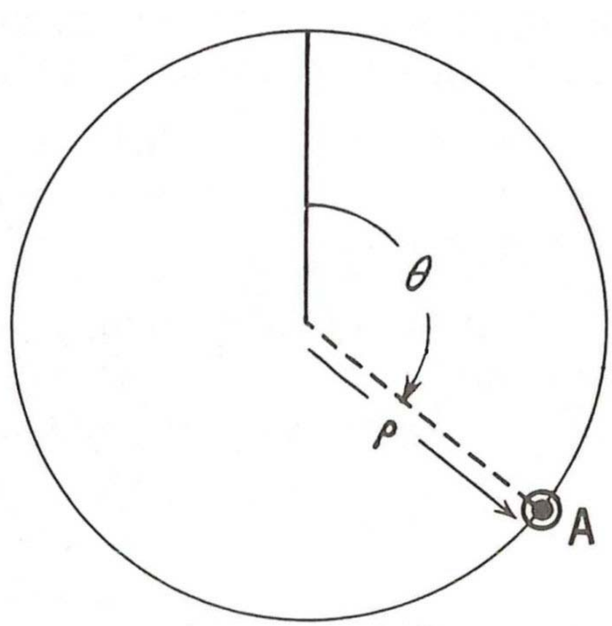


Figure 135b. Polar coordinates.

Spherical coordinates are used to define a position on the surface of a sphere or spheroid by indicating angular distance from a primary great circle and a reference secondary great circle. Examples used in navigation are latitude and longitude, altitude and azimuth, and declination and hour angle.

TRIGONOMETRY

136. Definitions

Trigonometry deals with the relations among the angles and sides of triangles. **Plane trigonometry** deals with plane triangles, those on a plane surface. **Spherical trigonometry** deals with spherical triangles, which are drawn on the surface of a sphere. In navigation, the common methods of celestial sight reduction use spherical triangles on the surface of the Earth. For most navigational purposes, the Earth is assumed to be a sphere, though it is somewhat flattened.

137. Angular Measure

A circle may be divided into 360 **degrees** (°), which is the **angular length** of its circumference. Each degree may be divided into 60 **minutes** ('), and each minute into 60 **seconds** ("). The angular measure of an arc is usually expressed in these units. By this system a right angle or quadrant has 90° and a straight angle or semicircle 180°. In marine navigation, altitudes, latitudes, and longitudes are usually expressed in degrees, minutes, and tenths (27°14.4'). Azimuths are usually expressed in degrees and tenths (164.7°). The system of degrees, minutes, and seconds indicated above is the **sexagesimal system**. In the **centesimal system,** used chiefly in France, the circle is divided into 400 **centesimal degrees** (sometimes called **grades**) each of which is divided into 100 centesimal minutes of 100 **centesimal seconds** each.

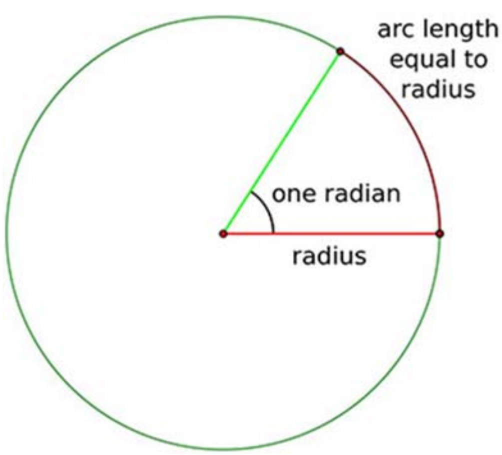


Figure 137. Image depicting one radian.

A **radian** is the angle subtended at the center of a circle by an arc having a linear length equal to the radius of the circle. A radian is equal to 57.2957795131° approximately, or 57°17'44.80625" approximately. The radian is sometimes used as a unit of angular measure. See Figure 137. A circle (360°) = 2π radians, a semicircle (180°) = π radians, a right angle measure (90°) = $\frac{\pi}{2}$ radians, and 1' = 0.0002908882 radians approximately.The length of the arc of a circle is equal to the radius multiplied by the angle subtended in radians.

138. Trigonometric Functions

Trigonometric functions are the various proportions or ratios of the sides of a plane right triangle, defined in relation to one of the acute angles. In Figure 138a, let θ be any acute angle. From any point R on line OA, draw a line perpendicular to OB at F. From any other point R' on OA, draw a line perpendicular to OB at F'. Then triangles OFR and OF'R' are similar right triangles because all their corresponding *angles* are equal. Since in any pair of similar triangles the ratio of any two sides of one triangle is equal to the ratio of the corresponding two sides of the other triangle.

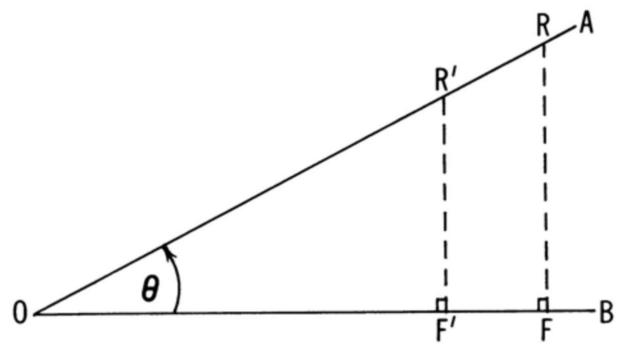


Figure 138a. Similar right triangles.

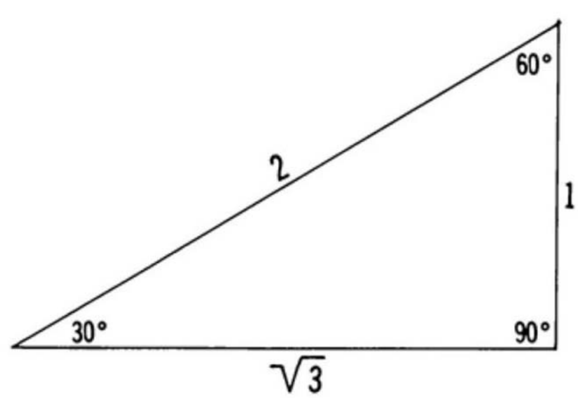


Figure 138b. *Numerical relationship of sides of a 30°-60°-90° triangle.*

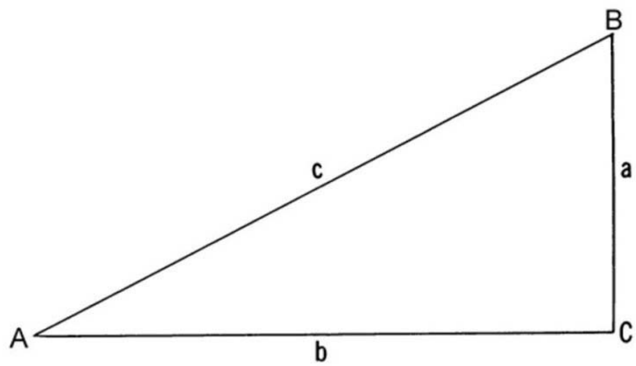


Figure 138c. A right triangle.

$$\frac{RF}{OF} = \frac{R'F'}{OF'},\ \frac{RF}{OR} = \frac{R'F'}{OR'},\ \text{and}\ \frac{OF}{OR} = \frac{OF'}{OR'}$$

No matter where the point R is located on OA, the ratio between the lengths of any two sides in the triangle OFR has a constant value. Hence, for any value of the acute angle θ, there is a fixed set of values for the ratios of the various sides of the triangle. These ratios are defined as follows:

$$\text{sine } \theta = \sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\text{cosine } \theta = \cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\text{tangent } \theta = \tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$\text{cosecant } \theta = \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\text{secant } \theta = \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\text{cotangent } \theta = \cot\theta = \frac{\text{side adjacent}}{\text{side opposite}}$$

Of these six principal functions, the second three are the reciprocals of the first three; therefore

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

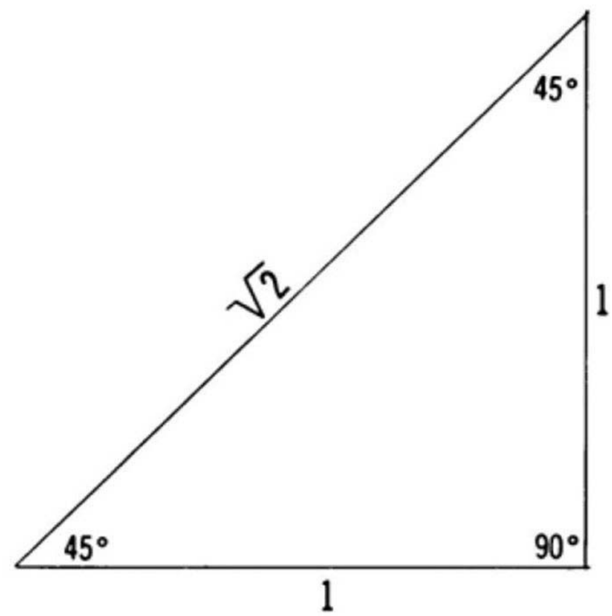


Figure 138d. Numerical relationship of sides of a 45°- 45° - 90° triangle.

In Figure 138c, *A*, *B*, and *C* are the angles of a plane right triangle, with the right angle at *C*. The sides are labeled *a*, *b*, *c*, with opposite angles labeled *A*, *B*, and *C* respectively.

The six principal trigonometric functions of angle *B* are:

$$\sin B = \frac{b}{c} = \cos A = \cos(90° - B)$$

$$\cos B = \frac{a}{c} = \sin A = \sin(90° - B)$$

$$\tan B = \frac{b}{a} = \cot A = \cot(90° - B)$$

$$\cot B = \frac{a}{b} = \tan A = \tan(90° - B)$$

$$\sec B = \frac{c}{a} = \csc A = \csc(90° - B)$$

$$\csc B = \frac{c}{b} = \sec A = \sec(90° - B)$$

Function	*30°*	*45°*	*60°*
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$	$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$
cosine	$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$	$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
tangent	$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$	$\frac{1}{1} = 1$	$\frac{\sqrt{3}}{1} = \sqrt{3}$
cotangent	$\frac{\sqrt{3}}{1} = \sqrt{3}$	$\frac{1}{1} = 1$	$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$
secant	$\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{1} = 2$
cosecant	$\frac{2}{1} = 2$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

Table 138e. Values of various trigonometric functions for angles 30°, 45°, and 60°.

Since *A* and *B* are *complementary*, these relations show that the sine of an angle is the cosine of its complement, the tangent of an angle is the cotangent of its complement, and the secant of an angle is the cosecant of its complement. Thus, the **co**-function of an angle is the function of its complement.

$$\sin(90° - A) = \cos A$$

$$\cos(90° - A) = \sin A$$

$$\tan(90° - A) = \cot A$$

$$\cot(90° - A) = \tan A$$

$$\sec(90° - A) = \csc A$$

$$\csc(90° - A) = \sec A$$

Certain additional relations are also classed as trigonometric functions:

versed sine θ = versine θ = vers θ = ver θ = 1- cos θ
versed cosine θ = coversed sine θ
(therefore) coversed sine θ = coversine θ
(therefore) coversine θ = covers θ
(therefore) covers θ = cov θ
(therefore) cov θ =1 - sin θ
haversine θ = hav $\theta = 1/2$ ver $\theta = (1/2)(1 - \cos\theta)$.

The numerical value of a trigonometric function is sometimes called the **natural function** to distinguish it from the logarithm of the function, called the **logarithmic function**. Numerical values of the six principal functions are given at 1' intervals in Table 2 - Natural Trigonometric Functions. Logarithms are given at the same intervals in Table 3 - Common Logarithms of Trigonometric Functions.

Since the relationships of 30°, 60°, and 45° right triangles are as shown in Figure 138c and Figure 138b, certain values of the basic functions can be stated exactly, as shown in Table 138e.

All trigonometric functions can be shown as lengths of lines in a unit circle. See Figure 138f for a depiction of the following equations:

$$\sin\theta = RF$$
$$\cot\theta = AB$$
$$\cos\theta = OF$$
$$\sec\theta = OD$$
$$\tan\theta = DE$$
$$\csc\theta = OA$$
$$\text{ver}\,\theta = FE$$
$$\text{cov}\,\theta = BC.$$

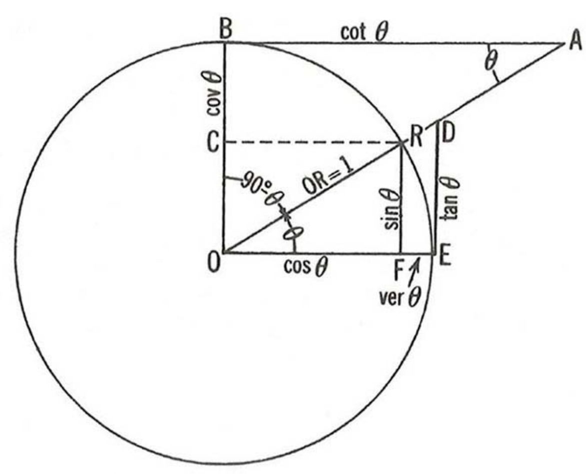


Figure 138f. Line definitions of trigonometric functions.

139. Functions in Various Quadrants

To make the definitions of the trigonometric functions more general to include those angles greater than 90°, the functions are defined in terms of the rectangular Cartesian coordinates of point R of Figure 138a, due regard being giv-

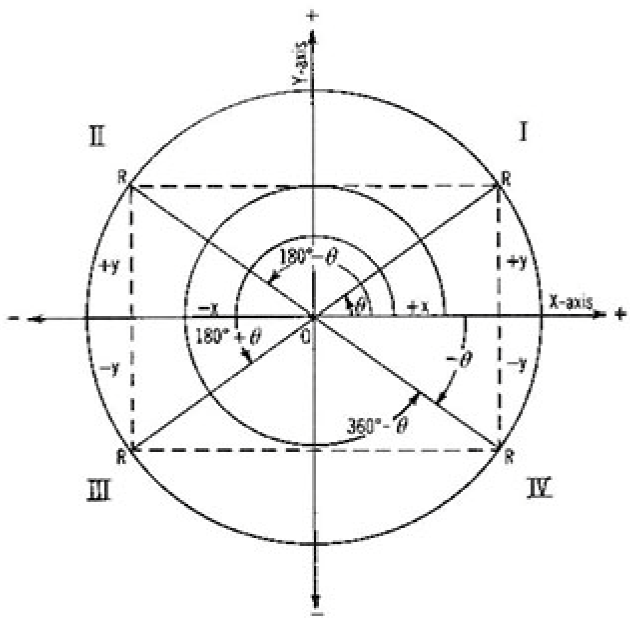


Figure 139a. The functions in various quadrants, mathematical convention.

en to the sign of the function. In Figure 139a, OR is assumed to be a *unit* radius. By convention the sign of OR is always positive. This radius is imagined to rotate in a counterclockwise direction through 360° from the horizontal position at 0°, the positive direction along the X axis. Ninety degrees (90°) is the positive direction along the Y axis. The angle between the original position of the radius and its position at any time increases from 0° to 90° in the *first quadrant* (I), 90° to 180° in the *second quadrant* (II), 180° to 270° in the *third quadrant* (III), and 270° to 360° in the *fourth quadrant* (IV).

The numerical value of the sine of an angle is equal to the projection of the unit radius on the Y–axis. According to the definition given in Section138, the sine of angle in the first quadrant of Figure 139a is $\frac{+y}{+OR}$. If the radius OR is equal to one, sin θ =+y. Since +y is equal to the projection of the unit radius OR on the Y axis, the sine function of an angle in the first quadrant defined in terms of rectangular Cartesian coordinates does not contradict the definition in Section 138. In Figure 139a,

$$\sin\theta = +y$$

$$\sin(180°-\theta) = +y = \sin\theta$$

$$\sin(180°+\theta) = -y = -\sin\theta$$

$$\sin(360°-\theta) = -y = \sin(-\theta) = -\sin\theta$$

The numerical value of the cosine of an angle is equal to the projection of the unit radius on the X axis. In Figure 139a,

$$\cos\theta = +x$$
$$\cos(180°-\theta) = -x = -\cos\theta$$
$$\cos(180°+\theta) = -x = -\cos\theta$$
$$\cos(360°-\theta) = +x = \cos(-\theta) = \cos\theta$$

The numerical value of the tangent of an angle is equal to the ratio of the projections of the unit radius on the Y and X axes. In Figure 139a,

$$\tan\theta = \frac{+y}{+x}$$

$$(180°-\theta) = \frac{+y}{-x} = -\tan\theta$$

$$\tan(180°+\theta) = \frac{-y}{-x} = \tan\theta$$

$$\tan(360°-\theta) = \frac{-y}{+x} = \tan(-\theta) = -\tan\theta$$

The cosecant, secant, and cotangent functions of angles in the various quadrants are similarly determined:

$$\csc\theta = \frac{1}{+y}$$

$$\csc(180°-\theta) = \frac{1}{+y} = \csc\theta$$

$$\csc(180°+\theta) = \frac{1}{-y} = -\csc\theta$$

$$\csc(360°-\theta) = \frac{1}{-y} = \csc(-\theta) = -\csc\theta$$

$$\sec\theta = \frac{1}{+x}$$

$$\sec(180°-\theta) = \frac{1}{-x} = -\sec\theta$$

$$\sec(180°+\theta) = \frac{1}{-x} = -\sec\theta$$

$$\sec(360°-\theta) = \frac{1}{+x} = \sec(-\theta) = \sec\theta$$

$$\cot\theta = \frac{+x}{+y}$$

$$\cot(180°-\theta) = \frac{-x}{+y} = -\cot\theta$$

$$\cot(180°+\theta) = \frac{-x}{-y} = \cot\theta$$

$$\cot(360°-\theta) = \frac{+x}{-y} = \cot(-\theta) = -\cot\theta.$$

The signs of the functions in the four different quadrants are shown below in Table 139b.

	I	II	III	IV
sine and cosecant	+	+	-	-
cosine and secant	+	-	-	+
tangent and cotangent	+	-	+	-

Table 139b. Signs of trigonometric functions by quadrants.

These relationships are shown in Table 139c and graphically in Figure 139d through Figure 139g.

	I	II	III	IV
sin	0 to +1	+1 to 0	0 to –1	–1 to 0
csc	+∞ to +1	+1 to 0	–∞ to –1	–1 to –∞
cos	+1 to 0	0 to –1	–1 to 0	0 to +1
sec	+1 to +∞	–∞ to –1	–1 to –∞	+∞ to +1
tan	0 to +∞	–∞ to 0	0 to +∞	–∞ to 0
cot	+∞ to 0	0 to –∞	+∞ to 0	0 to –∞

Table 139c. Values of trigonometric functions in various quadrants.

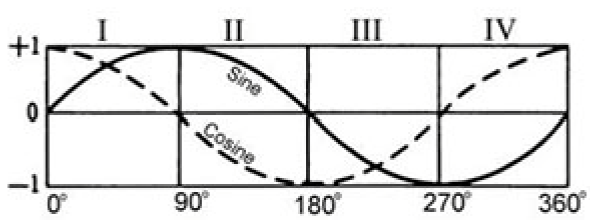


Figure 139d. Sine and cosine functions in various quadrants.

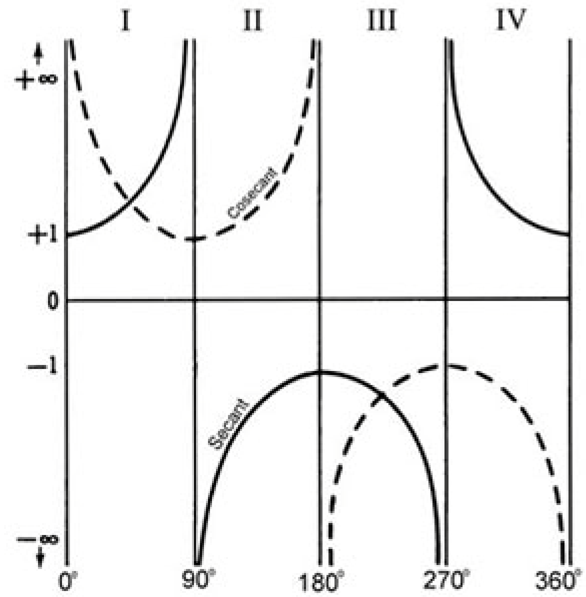


Figure 139e. Secant and cosecant functions in various quadrants.

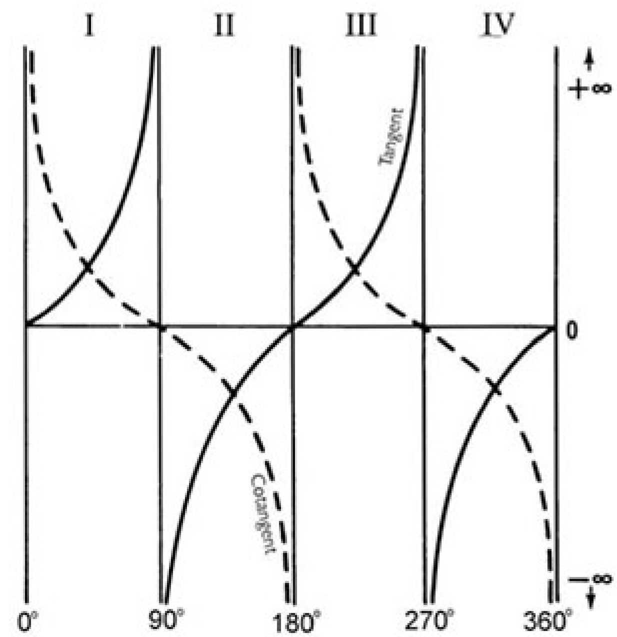


Figure 139f. Tangent and cotangent functions in various quadrants.

The numerical values vary by quadrant as shown above.

As shown in Figure 139a and Table 139b, the sign (+ or -) of the functions varies with the quadrant of an angle. In Figure 139a radius OR is imagined to rotate in a counter-

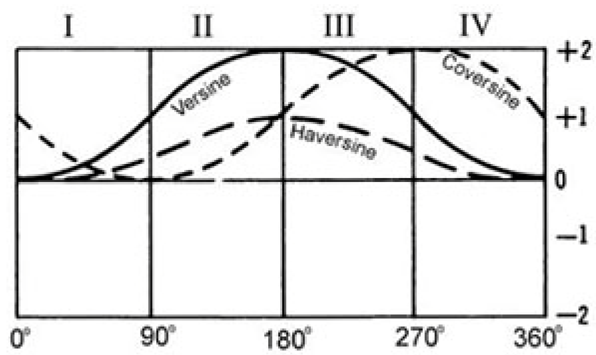


Figure 139g. Versine, coversine and haversine functions in various quadrants.

clockwise direction through 360° from the horizontal position at 0°. This is the **mathematical convention**. In Figure 139h this concept is shown in the usual **navigational convention** of a compass rose, starting with 000° at the top and rotating clockwise. In either diagram the angle θ between the original position of the radius and its position at any time increases from 0° to 90° in the *first quadrant* (I), 90° to 180° in the *second quadrant* (II), 180° to 270° in the *third quadrant* (III), and 270° to 360° in the *fourth quadrant* (IV). Also in either diagram, 0° is the positive direction along the X-axis. Ninety degrees (90°) is the positive direction along the Y-axis. Therefore, the projections of the unit radius OR on the X- and Y-axes, as appropriate, produce the same values of the trigonometric functions.

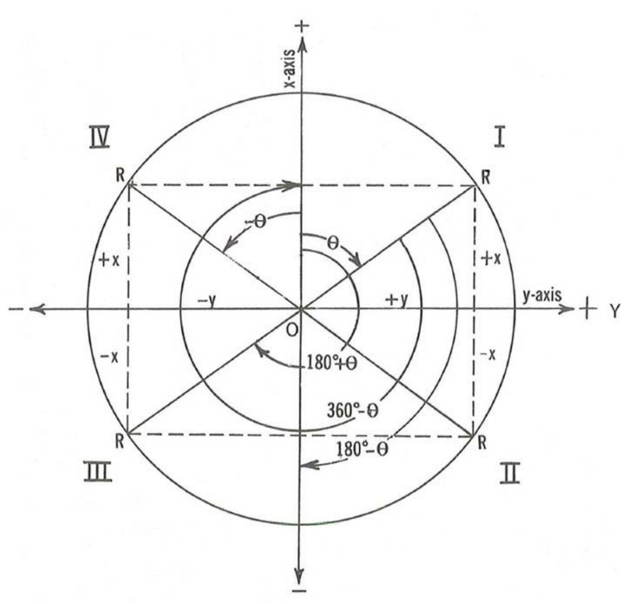


Figure 139h. The functions in various quadrants.

A negative angle (-θ) is an angle measured in a clockwise direction (mathematical convention) or in a direction opposite to that of a positive angle. The functions of a negative angle and the corresponding functions of a positive angle are as follows:

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan-\theta = -\tan\theta$$
$$\tan(-\theta) = \tan(360° - \theta)$$

140. Trigonometric Identities

A **trigonometric identity** is an equality involving trigonometric functions of θ which is true for all values of θ, except those values for which one of the functions is not defined or for which a denominator in the equality is equal to zero. The **fundamental identities** are those identities from which other identities can be derived.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

141. Reduction Formulas

$$\sin(90° - \theta) = \cos\theta \qquad \csc(90° - \theta) = \sec\theta$$
$$\cos(90° - \theta) = \sin\theta \qquad \sec(90° - \theta) = \csc\theta$$
$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\sin(90+\theta) = \cos\theta \qquad \csc(90+\theta) = \sec\theta$$
$$\cos(90+\theta) = -\sin\theta \qquad \sec(90+\theta) = -\csc\theta$$
$$\tan(90+\theta) = -\cot\theta \qquad \cot(90+\theta) = -\tan\theta$$

$\sin(180°+\theta) = -\sin\theta$ $\quad \csc(180°+\theta) = -\csc\theta$

$\cos(180°+\theta) = -\cos\theta$ $\quad \sec(180°+\theta) = -\sec\theta$

$\tan(180°+\theta) = \tan\theta$ $\quad \cot(180°+\theta) = \cot\theta$

$\sin(360° - \theta) = -\sin\theta$ $\quad \csc(360° - \theta) = -\csc\theta$

$\cos(360° - \theta) = \cos\theta$ $\quad \sec(360° - \theta) = \sec\theta$

$\tan(360° - \theta) = -\tan\theta$ $\quad \cot(360° - \theta) = -\cot\theta$

142. Inverse Trigonometric Functions

An angle having a given trigonometric function may be indicated in any of several ways. Thus, $\sin y = x$, $y = \text{arc}\sin x$, and $y = \sin^{-1} x$ have the same meaning. The superior "–1" is not an exponent in this case. In each case, y is "the angle whose sine is x." In this case, y is the **inverse sine** of x. Similar relationships hold for all trigonometric functions.

SOLVING TRIANGLES

Solution of triangles. A triangle is composed of six parts: three angles and three sides. The angles may be designated *A*, *B*, and *C*; and the sides opposite these angles as *a*, *b*, and *c*, respectively. In general, when any three parts are known, the other three parts can be found, unless the known parts are the three angles of a plane triangle.

143. Right Plane Triangles

In a **right plane triangle** it is only necessary to substitute numerical values in the appropriate formulas representing the basic trigonometric functions and solve. Thus, if *a* and *b* are known,

$$\tan A = \frac{a}{b}$$
$$B = 90° - A$$
$$c = a \csc A$$

Similarly, if *c* and *B* are given,

$$A = 90° - B$$
$$a = c \sin A$$
$$b = c \cos A$$

144. Oblique Plane Triangles

When solving an **oblique plane triangle**, it is often desirable to draw a rough sketch of the triangle approximately to scale, as shown in Figure 144. The following laws are helpful in solving such triangles:

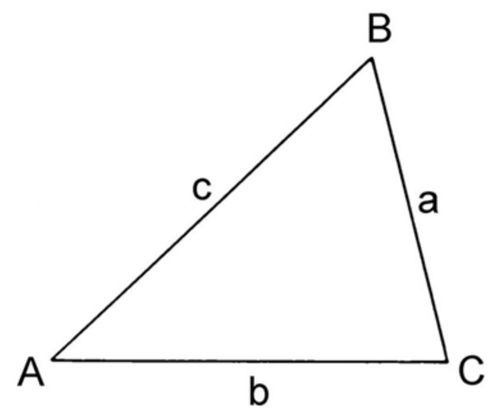


Figure 144. An oblique plane triangle.

Known	*To find*	*Formula*	*Comments*
a, b, c	*A*	$\cos A = \dfrac{c^2 + b^2 - a^2}{2bc}$	Cosine law
a, b, A	*B*	$\sin B = \dfrac{b \sin A}{a}$	Sine law. Two solutions if *b*>*a*
	C	$C = 180° - (A + B)$	$A + B + C = 180°$
	c	$c = \dfrac{a \sin C}{\sin A}$	Sine law
a, b, C	*A*	$\tan A = \dfrac{a \sin C}{b - a \cos C}$	

Table 144. Formulas for solving oblique plane triangles.

Known	*To find*	*Formula*	*Comments*
	B	$B = 180° - (A + C)$	$A + B + C = 180°$
	c	$c = \dfrac{a \sin C}{\sin A}$	Sine law
a, A, B	b	$b = \dfrac{a \sin B}{\sin A}$	Sine law
	C	$C = 180° - (A + B)$	$A + B + C = 180°$
	c	$c = \dfrac{a \sin C}{\sin A}$	Sine law

Table 144. Formulas for solving oblique plane triangles.

Law of sines: $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines: $$a^2 = b^2 + c^2 - 2bc \cos A.$$

The unknown parts of oblique plane triangles can be computed by the formulas in Table 144, among others. By reassignment of letters to sides and angles, these formulas can be used to solve for all unknown parts of oblique plane triangles.

SPHERICAL TRIGONOMETRY

145. Napier's Rules

Right spherical triangles can be solved with the aid of **Napier's Rules of Circular Parts**. If the right angle is omitted, the triangle has five parts: two angles and three sides, as shown in Figure 145a. Since the right angle is already known, the triangle can be solved if any two other parts are known. If the two sides forming the right angle, and the *complements* of the other three parts are used, these elements (called "parts" in the rules) can be arranged in five sectors of a circle in the same order in which they occur in the triangle, as shown in Figure 145b. Considering any part as the middle part, the two parts nearest it in the diagram are considered the adjacent parts, and the two farthest from it the opposite parts.

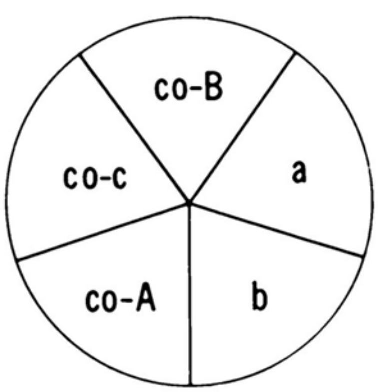


Figure 145b. Diagram for Napier's Rules of Circular Parts.

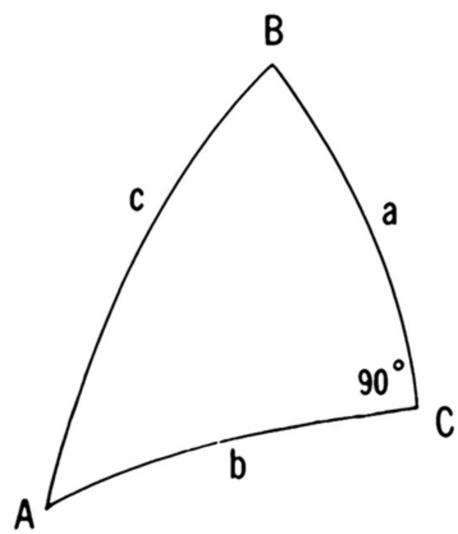


Figure 145a. Parts of a right spherical triangle as used in Napier's rules.

The following rules apply:

Napier's Rules state: The sine of a middle part equals the product of (1) the tangents of the adjacent parts or (2) the cosines of the opposite parts.

In the use of these rules, the co-function of a complement can be given as the function of the element. Thus, the cosine of co–A is the same as the sine of A. From these rules the following formulas can be derived:

$$\sin a = \tan b \ \cot B = \sin c \ \sin A$$
$$\sin b = \tan a \ \cot A = \sin c \ \sin B$$
$$\cos c = \cot A \ \cot B = \cos a \ \cos b$$
$$\cos A = \tan b \ \cot c = \cos a \ \sin B$$
$$\cos B = \tan a \ \cot c = \cos b \ \sin A$$

1. An oblique angle and the side opposite are in the same quadrant.
2. Side c (the hypotenuse) is less then 90° when a and b are in the same quadrant, and more than 90° when a and b are in different quadrants.

If the known parts are an angle and its opposite side, two solutions are possible.

A **quadrantal spherical triangle** is one having one side of 90°. A **biquadrantal spherical triangle** has two sides of 90°. A **triquadrantal spherical triangle** has three sides of 90°. A biquadrantal spherical triangle is isosceles and has two right angles opposite the 90° sides. A triquadrantal spherical triangle is equilateral, has three right angles, and bounds an octant (one–eighth) of the surface of the sphere. A quadrantal spherical triangle can be solved by Napier's rules provided any two elements in addition to the 90° side are known. The 90° side is omitted and the other parts are arranged in order in a five–sectored circle, using the complements of the three parts farthest from the 90° side. In the case of a quadrantal triangle, rule 1 above is used, and rule 2 restated: angle C (the angle opposite the side of 90°) is *more* than 90° when A and B are in the same quadrant, and *less* than 90° when A and B are in different quadrants. If the rule requires an angle of more than 90° and the solution produces an angle of less than 90°, subtract the solved angle from 180°.

146. Oblique Spherical Triangles

An **oblique spherical triangle** can be solved by dropping a perpendicular from one of the apexes to the opposite side, subtended if necessary, to form two right spherical triangles. It can also be solved by the following formulas in Table 146, reassigning the letters as necessary.

Known	*To find*	*Formula*	*Comments*
a, b, C	A	$\tan \mathrm{A} = \dfrac{\sin D \ \tan \mathrm{C}}{\sin\ (b - D)}$	$\tan D = \tan a \ \cos \mathrm{C}$
	B	$\sin \mathrm{B} = \dfrac{\sin \mathrm{C} \ \sin b}{\sin c}$	
c, A, B	C	$\cos \mathrm{C} = \sin \mathrm{A} \ \sin \mathrm{B} \ \cos c - \cos \mathrm{A} \ \cos \mathrm{B}$	
	a	$\tan a = \dfrac{\tan c \ \sin E}{\sin\ (\mathrm{B} + E)}$	$\tan E = \tan \mathrm{A} \ \cos c$
	b	$\tan b = \dfrac{\tan c \sin F}{\sin\ (\mathrm{A} + F)}$	$\tan F = \tan \mathrm{B} \ \cos c$
a, b, A	c	$\sin(c + G) = \dfrac{\cos a \ \sin G}{\cos b}$	$\cot G = \cos \mathrm{A} \ \tan b$ Two solutions
	B	$\sin \mathrm{B} = \dfrac{\sin \mathrm{A} \sin b}{\sin \mathrm{a}}$	Two solutions
	C	$\sin(\mathrm{C} + H) = \sin H \ \tan b \ \cot a$	$\tan H = \tan \mathrm{A} \ \cos b$ Two solutions
a, A, B	C	$\sin(\mathrm{C} - K) = \dfrac{\cos \mathrm{A} \sin K}{\cos \mathrm{B}}$	$\cot K = \tan \mathrm{B} \ \cos a$ Two solutions

Table 146. Formulas for solving oblique spherical triangles.

Known	*To find*	*Formula*	*Comments*
	b	$\sin b = \dfrac{\sin a \ \sin B}{\sin A}$	Two solutions
	c	$\sin(c - M) = \cot A \ \tan B \ \sin M$	$\tan M = \cos B \ \tan a$ Two solutions

Table 146. Formulas for solving oblique spherical triangles.

147. Other Useful Formulas

In addition to the fundamental trigonometric identities and reduction formulas given in Section 139, the following formulas apply to plane and spherical trigonometry:

Addition and Subtraction Formulas

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

Double-Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Half-Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}.$$

The following are useful formulas of spherical trigonometry:

Law of Cosines for Sides

$\cos a = \cos b \cos c + \sin b \sin c \cos A$

$\cos b = \cos c \cos a + \sin c \sin a \cos B$

$\cos c = \cos a \cos b + \sin a \sin b \cos C$

Law of Cosines for Angles

$\cos A = -\cos B \cos C + \sin B \sin C \cos a$

$\cos B = -\cos C \cos A + \sin C \sin A \cos b$

$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$

Law of Sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Napier's Analogies

$$\tan\frac{1}{2}(A + B) = \frac{\cos\frac{1}{2}(a - b)}{\cos\frac{1}{2}(a + b)}\cot\frac{1}{2}C$$

$$\tan\frac{1}{2}(A - B) = \frac{\sin\frac{1}{2}(a - b)}{\sin\frac{1}{2}(a + b)}\cot\frac{1}{2}C$$

$$\tan\frac{1}{2}(a + b) = \frac{\cos\frac{1}{2}(A - B)}{\cos\frac{1}{2}(A + B)}\tan\frac{1}{2}c$$

$$\tan\frac{1}{2}(a - b) = \frac{\sin\frac{1}{2}(A - B)}{\sin\frac{1}{2}(A + B)}\tan\frac{1}{2}c.$$

Five Parts Formulas

sin a cos B = cos b sin c - sin b cos c cos A

sin b cos C= cos c sin a - sin c cos a cos B

sin c cos A= cos a sin b - sin a cos b cos C.

Haversine Formulas

hav a = hav (b ~ c) + sin b sin c hav A

hav b = hav (a ~ c) + sin a sin c hav B

hav c = hav (a ~ b) + sin a sin b hav C

hav A= [hav a- hav (b ~ c)] csc b csc c

hav B = [hav b- hav (a ~ c)] csc a csc c

hav C = [hav c- hav (a ~ b)] csc a csc b.

148. Functions of a Small Angle

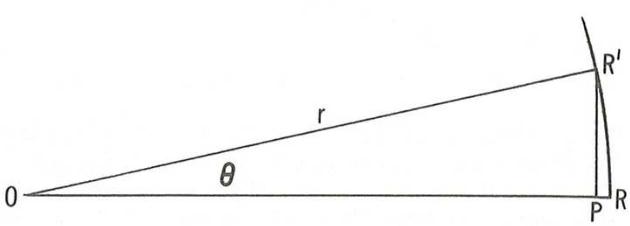


Figure 148. A small angle.

Functions of a small angle: In Figure 148, small angle θ, measured in radians, is subtended by the arc RR' of a circle. The radius of the circle is r, and $R'P$ is perpendicular to OR at P. Since the length of the arc of a circle is equal to the radius multiplied by the angle subtended in radians:

$$RR' = r \times \theta .$$

When θ is sufficiently small for $R'P$ to approximate RR',

$$\sin\theta = \theta$$

$$\text{since } \theta = \frac{RR'}{r} \text{ and } \sin\theta = \frac{R'P}{r} .$$

For small angles, it can also be shown that

$$\tan\theta = \theta .$$

If there are x minutes of arc (x') in a small angle of θ radians,

$$\sin x' = x \sin 1' .$$

Figure 148 also shows that when θ is small, OP is approximately equal to the radius. Therefore, cos θ can be taken as equal to 1.

Another approximation can be obtained if cos θ is expressed in terms of the half-angle:

$$\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$$

$$\cos\theta = 1 - 2\left(\frac{1}{2}\theta\right)^2$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2 .$$

CALCULUS

149. Calculus

Calculus is that branch of mathematics dealing with the rate of change of one quantity with respect to another.

A **constant** is a quantity which does not change. If a vessel is making good a course of 090°, the latitude does not change and is therefore a constant.

A **variable**, where continuous, is a quantity which can have an infinite number of values, although there may be limits to the maximum and minimum. Thus, from latitude 30° to latitude 31 ° there are an infinite number of latitudes, if infinitesimally small units are taken, but no value is less than 30° nor more than 31°. If two variables are so related that for every value of one there is a corresponding value of the other, one of the values is known as a **function** of the other. Thus, if speed is constant, the distance a vessel steams depends upon the elapsed time. Since elapsed time does not depend upon any other quantity, it is called an **independent variable**. The distance depends upon the elapsed time, and therefore is called a **dependent variable**. If it is required to find the time needed to travel any given distance at constant speed, distance is the independent variable and time is the dependent variable.

The principal processes of calculus are differentiation and integration.

150. Differentiation

Differentiation is the process of finding the rate of change of one variable with respect to another. If x is an independent variable, y is a dependent variable, and y is a function of x, this relationship may be written $y = f(x)$. Since for every value of x there is a corresponding value of y, the relationship can be plotted as a curve, Figure 150. In this figure, A and B are any two points on the curve, a short distance apart.

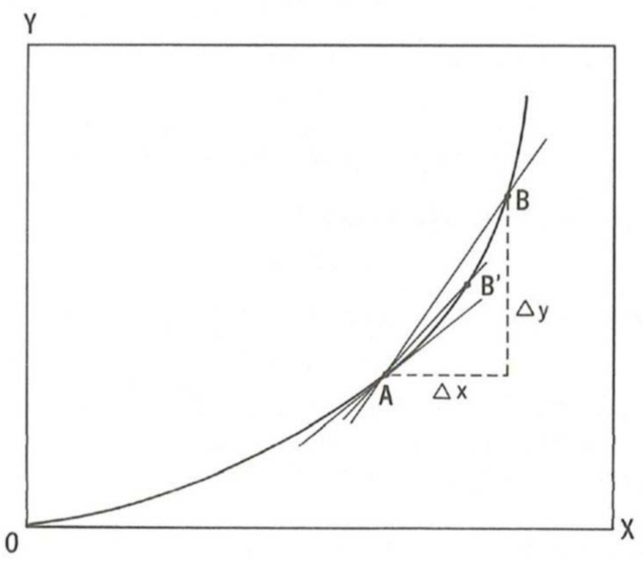


Figure 150. Differentiation.

The difference between the value of x at A and at B is Δx (delta x), and the corresponding difference in the value of y is Δy (delta y). The straight line through points A and B is a **secant** of the curve. It represents the rate of change between A and B for anywhere along this line the change of y is proportional to the change of x.

As B moves closer to A, as shown at B', both Δx and Δy become smaller, but at a different rate, and $\frac{\Delta y}{\Delta x}$ changes. This is indicated by the difference in the slope of the secant. Also, that part of the secant between A and B moves closer to the curve and becomes a better approximation of it. The limiting case occurs when B reaches A or is at an infinitesimal distance from it. As the distance becomes infinitesimal, both Δy and Δx become infinitely small, and are designated dy and dx, respectively. The straight line becomes tangent to the curve, and represents the rate of change, or slope, of the curve at that point. This is indicated by the expression $\frac{dy}{dx}$, called the **derivative** of y with respect to x.

The process of finding the value of the derivative is called **differentiation**. It depends upon the ability to connect x and y by an equation. For instance, if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$. If $n = 2$, $y = x^2$, and $\frac{dy}{dx} = 2x$. This is derived as follows: If point A on the curve is x, y; point B can be considered $x + \Delta x$, $y + \Delta y$. Since the relation $y = x^2$ is true anywhere on the curve, at B:

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2 .$$

Since $y = x^2$, and equal quantities can be subtracted from both sides of an equation without destroying the equality:

$$\Delta y = 2x\Delta x + (\Delta x)^2 .$$

Dividing by Δx:

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x .$$

As B approaches A, Δx becomes infinitesimally small, approaching 0 as a limit, Therefore $\frac{\Delta y}{\Delta x}$ approached $2x$ as a limit.

This can be demonstrated by means of a numerical example. Let $y = x^2$. Suppose at A, $x = 2$ and $y = 4$, and at B, $x = 2.1$ and $y = 4.41$. In this case $\Delta x = 0.1$ and $\Delta y = 0.41$, and

$$\frac{\Delta y}{\Delta x} = \frac{0.41}{0.1} = 4.1 .$$

From the other side of the equation:

$$2x + \Delta x = 2 \quad 2 + 0.1 = 4.1 .$$

If Δx is 0.01 and Δy is 0.0401, $\frac{\Delta y}{\Delta x} = 4.01$. If Δx is 0.001, $\frac{\Delta y}{\Delta x} = 4.001$; and if Δx is 0.0001, $\frac{\Delta y}{\Delta x} = 4.0001$. As Δx approaches 0 as a limit, $\frac{\Delta y}{\Delta x}$ approaches 4, which is therefore the value $\frac{dy}{dx}$. Therefore, at point A the *rate* of change of y with respect to x is 4, or y is increasing in value 4 times as fast as x.

151. Integration

Integration is the inverse of differentiation. Unlike the latter, however, it is not a direct process, but involves the recognition of a mathematical expression as the differential of a known function. The function sought is the **integral** of the given expression. Most functions can be differentiated, but many cannot be integrated.

Integration can be considered the summation of an infinite number of infinitesimally small quantities, between specified limits. Consider, for instance, the problem of finding an area below a specified part of a curve for which a mathematical expression can be written. Suppose it is desired to find the area $ABCD$ of Figure 151. If vertical lines are drawn dividing the area into a number of vertical strips, each Δx wide, and if y is the height of each strip at the midpoint of Δx, the area of each strip is approximately $y\Delta x$; and the approximate total area of all strips is the sum of the areas of the individual strips. This may be written $\sum_{x1}^{x2} y\Delta x$, meaning the sum of all $y\ \Delta x$ values between x_1 and x_2. The symbol Σ is the Greek letter *sigma*, the equivalent of the English *S*. If Δx is made progressively smaller, the sum of the small areas becomes ever closer to the true total area. If Δx becomes infinitely small, the summation expression is written $\int_{x1}^{x2} y dx$, the symbol dx denoting an infinitely small Δx. The symbol $\int$, called the "integral sign," is a distorted *S*.

An expression such as $\int_{x1}^{x2} y dx$ is called a **definite integral** because limits are specified (x_1 and x_2). If limits are not specified, as in $\int y dx$, the expression is called an **indefinite integral**.

A navigational application of integration is the finding of meridional parts, Table 6. The *rate* of change of meridional parts with respect to latitude changes progressively. The formula given in the explanation of the table is the equivalent of an integral representing the sum of the meridional parts from the equator to any given latitude.

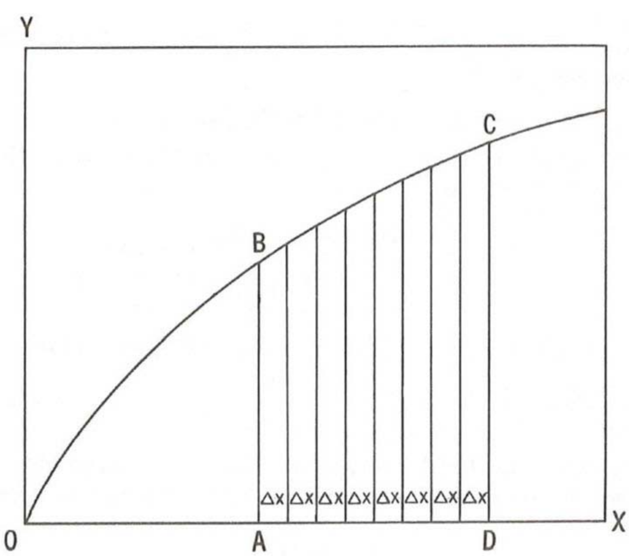


Figure 151. Integration.

152. Differential Equations

An expression such as *dy* or *dx* is called a **differential**. An equation involving a differential or a derivative is called a **differential equation**.

As shown in Section 150, if $y = x^2$, $\frac{dy}{dx} = 2x$. Neither *dy* nor *dx* is a finite quantity, but both are limits to which Δy and Δx approach as they are made progressively smaller. Therefore $\frac{dy}{dx}$ is merely a ratio, the limiting value of $\frac{\Delta y}{\Delta x}$, and not one finite number divided by another. However, since the ratio is the same as would be obtained by using finite quantities, it is possible to use the two differentials *dy* and *dx* independently in certain relationships. Differential equations involve such relationships.

Other examples of differential equations are:

$d \sin x = \cos x\ dx$ $\quad d \csc x = - \cot x \csc x\ dx$

$d \cos x = - \sin x\ dx$ $\quad d \sec x = \tan x \sec x\ dx$

$d \tan x = \sec^2 x\ dx$ $\quad d \cot x = - \csc^2 x\ dx.$

Some differential equations indicating the variations in the astronomical triangle are:

$dh = - \cos L \sin Z\ dt$; L and d constant

$dh = \cos Z\ dL$; d and t constant

$dh = - \cos h \tan M\ dZ$; L and d constant

$dZ = - \sec L \cot t\ dL$; d and h constant

$dZ = \tan h \sin Z\ dL$; d and t constant

$dt = - \sec L \cot Z\ dL$; d and h constant

$dZ = \cos d \sec h \cos M\ dt$; L and d constant

$dd = \cos d \tan M\ dt$; L and h constant

$dd = \cos L \sin t\ dZ$; L and h constant,

where h is the altitude, L is the latitude, Z is the azimuth angle, d is the declination, t is the meridian angle, and M is the parallactic angle.

CHAPTER 2

INTERPOLATION

FINDING THE VALUE BETWEEN TABULATED ENTRIES

200. Introduction

When one quantity varies with changing values of a second quantity, and the mathematical relationship of the two is known, a curve can be drawn to represent the values of one corresponding to various values of the other. To find the value of either quantity corresponding to a given value of the other, one finds that point, on the curve defined by the given value, and reads the answer on the scale relating to the other quantity. This assumes, of course, that for each value of one quantity, there is only one value of the other quantity.

Information of this kind can also be tabulated. Each entry represents one point on the curve. The finding of value *between* tabulated entries is called **interpolation**. The extending of tabulated values to find values *beyond* the limits of the table is called **extrapolation**.

Thus, the *Nautical Almanac* tabulates values of declination of the sun for each hour of Coordinated Universal Time (UTC) or Universal Time (UT). The finding of declination for a time between two whole hours requires interpolation. Since there is only one entering **argument** (in this case UT), **single interpolation** is involved.

Table 11 gives the distance traveled in various times at certain speeds. In this table there are two entering arguments. If both given values are between tabulated values, **double interpolation** is needed.

In *Pub. No. 229*, azimuth angle varies with a change in any of the three variables: latitude, declination, and local hour angle. With intermediate values of all three, **triple interpolation** is needed.

Interpolation can sometimes be avoided. A table having a single entering argument can be arranged as a **critical table**. An example is the dip (height of eye) correction on the inside front cover of the *Nautical Almanac*. Interpolation is avoided through dividing the argument into intervals so chosen that successive intervals correspond to successive values of the required quantity, the respondent. For any value of the argument within these intervals, the respondent can be extracted from the table without interpolation. The lower and upper limits (critical values) of the argument correspond to half-way values of the respondent and, by convention, are chosen so that when the argument is equal to one of the critical values, the respondent corresponding to the preceding (upper) interval is to be used. Another way of avoiding interpolation would be to include every possible entering argument. If this were done for *Pub. No. 229*, interpolation being eliminated for declination only, and assuming declination values to 0'.1, the number of volumes would be increased from six to more than 3,600. If interpolation for meridian angle and latitude, to 0'.1, were also to be avoided, a total of more than 1,296,000,000 volumes would be needed. A more practical method is to select an assumed position to avoid the need for interpolation for two of the variables.

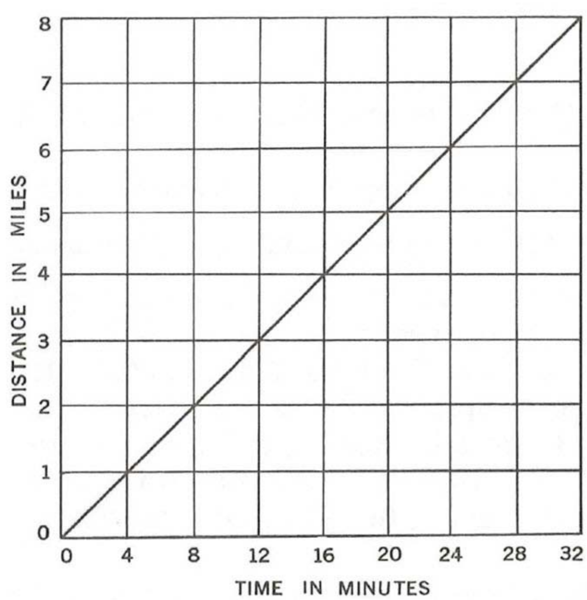


Figure 201a. Plot of D = t / 4.

201. Single Interpolation

The accurate determination of intermediate values requires knowledge of the nature of the change between tabulated values. The simplest relationship is linear, the change in the tabulated value being directly proportional to the change in the entering argument. Thus, if a vessel is proceeding at 15 knots, the distance traveled is directly proportional to the time, as shown in Figure 201a. The same information might be given in tabular form, as shown in table 201b. Mathematically, this relationship for 15 knots is written $D = \frac{15t}{60} = \frac{t}{4}$, where D is distance in nautical miles, and t is time in minutes.

In such a table, interpolation can be accomplished by simple proportion. Suppose, for example, that the distance is desired for a time of 15 minutes. It will be some value between 3.0 and 4.0 miles, because these are the distances for 12 and 16 minutes, respectively, the tabulated times on each side of the desired time.

Minutes	Miles
0	0.0
4	1.0
8	2.0
12	3.0
16	4.0
20	5.0
24	6.0
28	7.0
32	8.0

Table 201a. Table of D = t / 4

The proportion might be formed as follows:

$$\left.\begin{array}{rl} 3 & \left[\begin{array}{l}12\\15\end{array}\right. \\ & \;16 \end{array}\right]4$$

$$\left.\begin{array}{rl} x & \left[\begin{array}{l}3.0\\y\end{array}\right. \\ & \;4.0 \end{array}\right]1.0$$

$$\frac{3}{4} = \frac{x}{1.0}$$

$$x = \frac{3 \quad 1.0}{4} = 0.75 \text{ (0.8 to the nearest 0.1 mi.)}$$

$$y = 3.0 + x = 3.0 + 0.8 = 3.8 \text{ mi.}$$

A simple interpolation such as this should be performed mentally. During the four-minute interval between 12 and 16 minutes, the distance *increases* 1.0 mile from 3.0 to 4.0 miles. At 15 minutes, 3/4 of the interval has elapsed, and so the distance Increases 3/4 of 1.0 mile, or 0.75 mile, and is therefore 3.0+0.8=3.8, to the nearest 0.1 mile.

This might also have been performed by starting with 16 minutes, as follows:

$$\left.\begin{array}{rl} & \;12 \\ 1 & \left[\begin{array}{l}15\\16\end{array}\right. \end{array}\right]4$$

$$\left.\begin{array}{rl} & \;3.0 \\ (-)x & \left[\begin{array}{l}y\\4.0\end{array}\right. \end{array}\right]1.0$$

$$\frac{1}{4} = \frac{(-)x}{1.0}$$

$$x = (-)0.25 \text{ (-0.2 to the nearest 0.1 mi.)}$$

$$y = 4.0 - 0.2 = 3.8$$

Mentally, 15 is one quarter of the way from 16 to 12, and therefore the distance is 1/4 the way between 4.0 and 3.0, or 3.8.

This interpolation might have been performed by noting that if distance changes 1.0 mile in four minutes, it must change $\frac{1.0}{10} = 0.1$ mile in $\frac{4}{10} = 0.4$ minute, or 24 seconds.

This relationship can be used for mental interpolation in situations which might seem to require pencil and paper. Thus, if distance to the nearest 0.1 mile is desired for 13m 15s, the answer is 3.3 miles, determined as follows: The time 13^m15^s is 1^m15^s (1.2^m approx.) more than 12^m. If 1.2 is divided by 0.4, the quotient is 3, to the nearest whole number. Therefore, $3 \quad 0.1 = 0.3$ is added to 3, the tabulated value for 12 minutes. Alternatively, 13^m15^s is 2^m45^s (2.8m approx.) less than 16^m, and $2.8 \quad 0.4 = 7$, and therefore the interpolated value is $7 \quad 0.1 = 0.7$ *less* than 4, the tabulated value for 16^m. In either case, the interpolated value is 3.3 miles,

A common mistake in single interpolation is to apply the correction (*x*) with the wrong sign, particularly when it should be negative (-). This mistake can be avoided by always checking to be certain that the interpolated value lies between the two values used in the interpolation.

When the curve representing the values of a table is a straight line, as in a, the process of finding intermediate values in the manner described above is called **linear interpolation**. If tabulated values of such a line are exact (not approximations), as in Table 201a, the interpolation can be carried to any degree of precision without sacrificing accuracy. Thus, in 21.5 minutes the distance is $5.0 + \frac{1.5}{4} \quad 1.0 = 5.375$ miles. Similarly, for 29.9364 minutes the distance is $7.0 + \frac{1.9364}{4} \quad 1.0 = 7.4841$ miles, a value which has little or no significance in practical navigation. If one had occasion to find such a value, it could most easily be done by dividing the time, in minutes, by 4, since the distance increases at the rate of one mile each four minutes. This would be a case of avoiding interpolation by solving the equation connecting the two quantities. For a simple relationship such as that involved here, such a solution might be easier than interpolation.

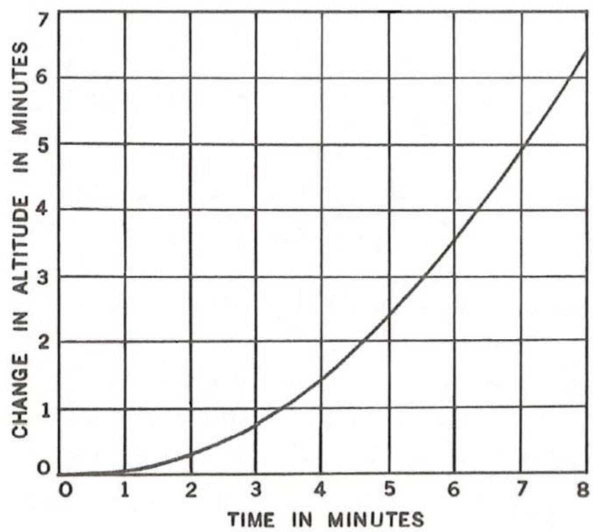


Figure 201b. Plot of altitude change = at^2.

Many of the tables of navigation are not linear. Consider Figure 201b. From Table 24 (Altitude Factors) it is found that for latitude 25° and declination 8°, same name, the variation of altitude in one minute of time from meridian transit (the altitude factor) is 6.0" (0.1'). For limited angular distance on each side of the celestial meridian, the change in altitude is approximately equal to at^2, where a is the altitude factor (Table 24) and t is the time in minutes from meridian transit. Figure 201b is the plot of change in altitude against time. The same information is shown in tabular form in Table 201c.

Minutes	*Miles*
0	0.0
1	0.1
2	0.4
3	0.9
4	1.6
5	2.5
6	3.6
7	4.9
8	6.4

Table 201c. Table of altitude change = at^2, where a=0.1'.

To be strictly accurate in interpolating in such a table, one should consider the curvature of the line. However, in most navigational tables the points on the curve selected for tabulation are sufficiently close that the portion of the curve between entries can be considered a straight line without introducing a significant error. This is similar to considering the line of position from a celestial observation as a part of the circle of equal altitude. Thus, to the nearest 0.1', the change of altitude for 3.4 minutes is $0.9' + (0.4 \quad 0.7') = 0.9' + 0.3' = 1.2'$. The correct value by solution of the formula is 1.156'. The value for 6.8 minutes is 4.6' by interpolation and 4.624' by computation.

Section 204 (Nonlinear Interpolation) addresses the nonlinear interpolation used when the curve representing tabular values under consideration is not a close approximation to a straight line. However, such instances are infrequent in navigation, and generally occur at a part of the navigation table that is not commonly used, or for which special provisions are made. For example, in *Pub. No. 229* nonlinear interpolation *may* be required only when the altitude is above 60°. Even when the altitude is above 60°, the need for nonlinear interpolation is infrequent. When it is needed, such fact is indicated by the altitude difference being printed in italic type followed by a small dot.

202. Double Interpolation

In a double-entry table it may be necessary to interpolate for each entering argument. Table 202a is an extract from Table 22 (amplitudes). If one entering argument is an exact tabulated value, the amplitude can be found by single interpolation. For instance, if latitude is 45° and declination is 21.8°, amplitude is $31.2° + \left(\frac{3}{5} \quad 0.8°\right) = 31.2° + 0.5° = 31.7°$. However, if neither entering argument is a tabulated value, double interpolation is needed. This may be accomplished in any of several ways:

Lat.	*Declination*	
	21.5°	22.0°
°	°	°
45	31.2	32.0
46	31.8	32.6

Table 202a. Excerpts from amplitude table.

"Horizontal" method. Use single interpolation for declination for each tabulated value of latitude, followed by single interpolation for latitude. Suppose latitude is 45.7° and declination is 21.8°. First, find the amplitude for latitude 45°, declination 21.8°, as above, 31.7°. Next, repeat the process for latitude 46°: $31.8° + \left(\frac{3}{5} \quad 0.8°\right) = 32.3°$. Finally, interpolate between 31.7° and 32.3° for latitude

45.7°: $31.7° + (0.7 \quad 0.6°) = 32.1°$. This is the equivalent of first inserting a new column for declination 21.8°, followed by single interpolation in this column, as shown in Table 202b.

Lat.	*Declination*		
	21.5°	*21.8°*	22.0°
°	°	°	°
45	31.2	*31.7*	32.0
45.7		***32.1***	
46	31.8	*32.3*	32.6

Table 202b. "Horizontal" method of double interpolation.

"Vertical" method. Use single interpolation for latitude for each tabulated value of declination, followed by single interpolation for declination. Consider the same example as above. First, find the amplitude for declination 21.5°, latitude 45.7°: $31.2° + (0.7° \quad 0.6°) = 31.6°$. Next, repeat the process for declination 22.0°: $32.0° + (0.7° \quad 0.6°) = 32.4°$. Finally, interpolate between 31.6° and 32.4° for declination 21.8°: $31.6° + \left(\frac{3}{5} \quad 0.8°\right) = 32.1°$. This is the equivalent of first inserting a new line for latitude 45.7°, followed by single interpolation in this line, as shown in Table 202c.

Lat.	*Declination*		
	21.5°	*21.8°*	22.0°
°	°	°	°
45	31.2		32.0
45.7	*31.6*	***32.1***	*32.4*
46	31.8		32.6

Table 202c. "Vertical" method of double interpolation.

"Combined" method. Select a tabulated "base" value, preferably that nearest the given tabulated entering arguments. Next, find the correction to be applied, with its sign, for single interpolation of this base value both horizontally and vertically. Finally, add these two corrections algebraically and apply the result, in accordance with its sign, to the base value, In the example given above, the base value is 32.6°, for declination 22.0° (21.8° is nearer 22.0° than 21.5°) and latitude 46° (45.7° is nearer 46° than 45°). The correction for declination is $\frac{2}{5} \quad (-)0.8° = (-)0.3°$. The correction for latitude is $0.3° \quad (-)0.6° = (-)0.2°$. The algebraic sum is $(-)0.3° + (-)0.2° = (-)0.5°$. The interpolated value is then $32.6° - 0.5° = 32.1°$. This is the method customarily used by navigators, however, it is also less precise. If more accuracy is required more tedium must be exercised using the horizontal or vertical methods.

203. Triple Interpolation

With three entering arguments, the process is similar to that for double interpolation. It would be possible to perform double interpolation for the tabulated value on each side of the given value of one argument, and then interpolate for that argument, but the method would be tedious. The only method commonly used by navigators is that of selecting base value and applying corrections.

204. Nonlinear Interpolation

When the curve representing the values of a table is nearly a straight line, or the portion of the curve under consideration is nearly a straight line, linear interpolation suffices. However, when the successive tabular values are so nonlinear that a portion of the curve under consideration is not a close approximation to a straight line, it is necessary to include the effects of second differences, and possibly higher differences, as well as first differences in the interpolation.

The plot of Table 204a data in Figure 204b indicates that the altitude does not change linearly between declination values of 51° and 52°. If the first difference only were used in the interpolation, the interpolated value of altitude would lie on the straight line between points on the curve for declination values of 51° and 52°.

LHA 38°, Lat. 45° (Same as Dec.)			
Dec.	ht (Tab. Hc)	*First Difference*	Second Difference
50°	64°08.2’		
		+2.8’	
51°	64°11.0’		-2.3’
		+0.5’	
52°	64°11.5’		-2.1’
		-1.6’	
53°	64°09.9’		

Table 204a. Data from Pub. No. 229.

If the altitude for declination 51°30' is obtained using only the **first difference**, i.e., the difference between successive tabular altitudes in this case, $Hc = 64°11.0' + \frac{30'}{60'} \quad 0.5' = 64°11.3'$. However, inspec-

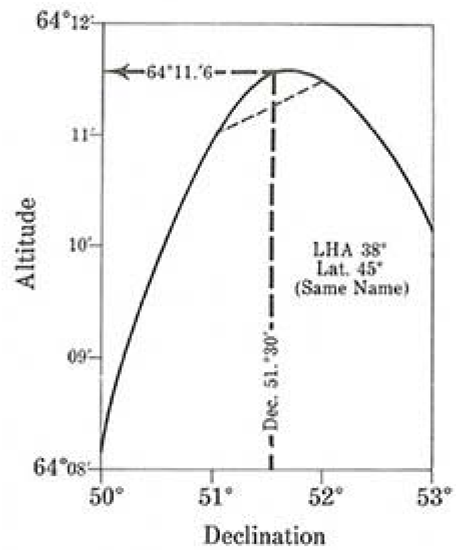


Figure 204b. Altitude curve.

tion of Figure 204b reveals that this interpolated altitude is 0.3' low. If the tabular data were such that the differences between successive first differences, the **second differences**, were nearly zero, interpolation using the first difference only would provide the correct altitude. In this case, however, second differences are significant and must be included in the interpolation.

Function	*First Difference*	Second Difference
f_{-2}		δ^2_{-2}
	$\delta_{-3/2}$	
f_{-1}		δ^2_{-1}
	$\delta_{-1/2}$	

Table 204c. Notation used with Bessel's Formula.

Function	*First Difference*	Second Difference
f_0		δ^2_0
	$\delta_{1/2}$	
f_{+1}		δ^2_1
	$\delta_{3/2}$	
f_{+2}		δ^3_2

Table 204c. Notation used with Bessel's Formula.

Table 204c shows the format and notation used to distinguish the various tabular quantities and differences when using Bessel's formula for the nonlinear interpolation. The quantities $f_{-2}, f_{-1}, f_0, f_{+1}, f_{+2}, f_{+3}$ represents represent successive tabular values.

Allowing for first and second differences only, Bessel's formula is stated as:

$$f_p = f_0 + p\delta_{1/2} + B_2\left(\delta^2_0 + \delta^2_1\right)$$

In this case, f_p is the computed altitude; f_0 is the tabular altitude; p is the fraction of the interval between tabular values of declination. The quantity B_2 is a function of p and is always negative. This coefficient is tabulated in Table 204d. The quantity $\left(\delta^2_0 + \delta^2_1\right)$ is the **double second difference (DSD)**, which is the sum of successive second differences.

Applying Bessel's formula to the data of Table 204a to obtain the altitude for a declination of 51°30',

$$f_p = f_0 + p\delta_{1/2} + B_2\left(\delta^2_0 + \delta^2_1\right)$$

$$Hc = 64°11.0' + \left(\frac{30'}{60'}\right)(0.5') + (-0.062)[-2.3' + (-2.1')]$$

$$Hc = 64°11.0' + 0.3' + 0.3' = 64°11.6'$$

p	B_2	p	B_2	p	B_2	p	B_2	p	B_2
0.0000	.000	0.1101	.025	0.2719	.050	0.7280	.049	0.8898	.024
.0020	.001	.1152	.026	.2809	.051	.7366	.048	.8949	.023
.0060	.002	.1205	.027	.2902	.052	.7449	.047	.9000	.022

Table 204d. Bessel's Coefficient B_2. In critical cases ascend. B_2 is always negative.

p	B_2	p	B_2	p	B_2	p	B_2	p	B_2
.0101	.003	.1258	.028	.3000	.053	.7529	.046	.9049	.021
.0142	.004	.1312	.029	.3102	.054	.7607	.045	.9098	.020
.0183	.005	.1366	.030	.3211	.055	.7683	.044	.9147	.019
.0225	.006	.1422	.031	.3326	.056	.7756	.043	.9195	.018
.0267	.007	.1478	.032	.3450	.057	.7828	.042	.9242	.017
.0309	.008	.1535	.033	.3585	.058	.7898	.041	.9289	.016
.0352	.009	.1594	.034	.3735	.059	.7966	.040	.9335	.015
.0395	.010	.1653	.035	.3904	.060	.8033	.039	.9381	.014
.0439	.011	.1713	.036	.4105	.061	.8098	.038	.9427	.013
.0483	.012	.1775	.037	.4367	.062	.8162	.037	.9472	.012
.0527	.013	.1837	.038	.5632	.061	.8224	.036	.9516	.011
.0572	.014	.1901	.039	.5894	.060	.8286	.035	.9560	.010
.0618	.015	.1966	.040	.6095	.059	.8346	.034	.9604	.009
.0664	.016	.2033	.041	.6264	.058	.8405	.033	.9647	.008
.0710	.017	.2101	.042	.6414	.057	.8464	.032	.9690	.007
.0757	.018	.2171	.043	.6549	.056	.8521	.031	.9732	.006
.0804	.019	.2243	.044	.6673	.055	.8577	.030	.9774	.005
.0852	.020	.2316	.045	.6788	.054	.8633	.029	.9816	.004
.0901	.021	.2392	.046	.6897	.053	.8687	.028	.9857	.003
.0950	.022	.2470	.047	.7000	.052	.8741	.027	.9898	.002
.1000	.023	.2550	.048	.7097	.051	.8794	.026	.9939	.001
.1050	.024	.2633	.049	.7190	.050	.8847	.025	0.9979	.000
0.1101		0.2719		0.7280		0.8898		1.0000	

Table 204d. Bessel's Coefficient B_2. In critical cases ascend. B_2 is always negative.

205. Interpolation Tables

A number of frequently used navigation tables are provided with auxiliary tables to assist in interpolation. Table 1 (Logarithms of Numbers) provides columns of "d" (difference between consecutive entries) and auxiliary "proportional parts" tables. The auxiliary table for the applicable difference "d" is selected and entered with the digit of the additional place in the entering argument. The value taken from the auxiliary table is *added* to the base value for the next *smaller* number from the main table. Suppose the logarithm (mantissa) for 32747 is desired. The base value for 3274 is 51508, and "d" is 13. The auxiliary table for 13 is entered with 7, and the correction is found to be 9. If this is added to 51508, the interpolated value is found to be 51517. This is the same result that would be obtained by subtracting 51508 from 51521 (the logarithm for 3275) to obtain 13, multiplying this by 0.7, and adding the result (9) to 51508.

Table 1 (Logarithms of Numbers) and Table 2 (Natural Trigonometric Functions) provide the difference between consecutive entries, but no proportional parts tables.

The *Nautical Almanac* "Increments and Corrections" are interpolation tables for the hourly entries of Greenwich Hour Angle (GHA) and declination. The increments are the products of the constant value used as the change of GHA in 1 hour and the fractional part of the hour. The corrections provide for the difference between the actual change of GHA in 1 hour and the constant value used. The corrections also provide the product of the change in declination in 1 hour and the fractional part of the hour.

The main part of the four-page interpolation table of *Pub. No. 229* is basically a multiplication table providing tabulations of:

$$\text{Altitude Difference} \quad \frac{\text{Declination Increment}}{60'}$$

The design of the table is such that the desired product must be derived from component parts of the altitude difference. The first part is a multiple of 10' (10', 20', 30', 40', or 50') of the altitude difference; the second part is the remainder in the range 0.0’ to 9.9’. For example, the component parts of altitude difference 44.3’ are 40' and 4.3’.

In the use of the first part of the altitude difference, the table arguments are declination increment (Dec. Inc.) and the integral multiple of 10' in the altitude difference, d. As shown in Figure 205a, the respondent is:

$$\text{Tens} \quad \frac{\text{Dec. Inc.}}{60'} .$$

In the use of the second part of the altitude difference, the interpolation table arguments are the nearest Dec. Inc. ending in 0.5’ and Units and Decimals. The respondent is:

$$\text{Units and Decimals} \quad \frac{\text{Dec. Inc.}}{60'} .$$

In computing the table, the values in the Tens part of the multiplication table were modified by small quantities varying from -0.042’ to +0.033’ before rounding to the tabular precision to compensate for any difference between the actual Dec. Inc. and the nearest Dec. Inc. ending in 0.5’ when using the Units and Decimals part of the table.

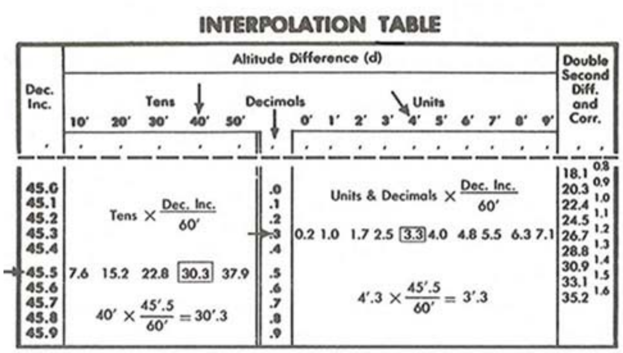
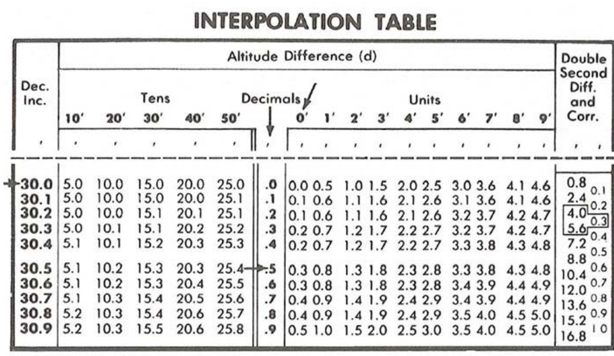


Figure 205a. Interpolation table.

Using the interpolation table shown in Figure 205b to obtain the altitude for 51°30' from the data of Table 204a (*Data from Pub. No. 229)*, the linear correction for the first difference (+0.5’) is +0.3’. This correction is extracted from the Units and Decimals block opposite the Dec. Inc. (30.0’). The correction for the **double second difference (DSD)** is extracted from the DSD subtable opposite the block in which the Dec. Inc. is found. The argument for entering this critical table is the DSD (-4.4’). The DSD correction is +0.3’. Therefore,

$$Hc = ht + \text{first difference correction} + \text{DSD correction}$$
$$= 64°11.0' + 0.3' + 0.3' = 64°11.6' .$$



INTERPOLATION TABLE

Dec. Inc.	Tens 10'	20'	30'	40'	50'	Decimals	Units 0'	1'	2'	3'	4'	5'	6'	7'	8'	9'	Double Second Diff.	and Corr.
30.0	5.0	10.0	15.0	20.0	25.0	.0	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.6	4.1	4.6	0.8	0.1
30.1	5.0	10.0	15.0	20.0	25.1	.1	0.1	0.6	1.1	1.6	2.1	2.6	3.1	3.6	4.1	4.6	2.4	0.2
30.2	5.0	10.0	15.1	20.1	25.1	.2	0.1	0.6	1.1	1.6	2.1	2.6	3.2	3.7	4.2	4.7	4.0	0.3
30.3	5.0	10.1	15.1	20.2	25.2	.3	0.2	0.7	1.2	1.7	2.2	2.7	3.2	3.7	4.2	4.7	5.6	0.4
30.4	5.1	10.1	15.2	20.3	25.3	.4	0.2	0.7	1.2	1.7	2.2	2.7	3.3	3.8	4.3	4.8	7.2	0.5
30.5	5.1	10.2	15.3	20.3	25.4	.5	0.3	0.8	1.3	1.8	2.3	2.8	3.3	3.8	4.3	4.8	8.8	0.6
30.6	5.1	10.2	15.3	20.4	25.5	.6	0.3	0.8	1.3	1.8	2.3	2.8	3.4	3.9	4.4	4.9	10.4	0.7
30.7	5.1	10.3	15.4	20.5	25.6	.7	0.4	0.9	1.4	1.9	2.4	2.9	3.4	3.9	4.4	4.9	12.0	0.8
30.8	5.2	10.3	15.4	20.6	25.7	.8	0.4	0.9	1.4	1.9	2.4	2.9	3.5	4.0	4.5	5.0	13.6	0.9
30.9	5.2	10.3	15.5	20.6	25.8	.9	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	15.2	1.0
																	16.8	

Figure 205b. Interpolation table.

More on Second Differences using Pub 229. The accuracy of linear interpolation usually decreases as the altitude increases. At altitudes above 60° it may be necessary to include the effect of second differences in the interpolation. When the altitude difference, d, is printed in italic type followed by a small dot, the second-difference correction may exceed 0.25’, and should normally be applied. The need for a second-difference correction is illustrated by the graph of Table 205c data in Figure 205d.

LHA 28°, Lat. 15° (Same as Dec.)			
Dec.	ht (Tab. Hc)	*First Difference*	Second Difference
15°	62°58.4’		
		+2.8’	
16°	63°01.2’		-2.0’
		+0.8’	
17°	63°02.0’		-2.1’
		-1.3’	
18°	63°00.7’		

Table 205c. Data from Pub. No. 229.

Other than graphically, the required correction for the effects of second differences is obtained from the appropriate subtable of the Interpolation Table. However, before the Interpolation Table can be used for this purpose, what is known as the double-second difference (DSD) must be formed.

Forming the Double-Second Difference (DSD). The double-second difference is the sum of two successive second differences. Although second differences are not tabulated, the DSD can be formed readily by subtracting,

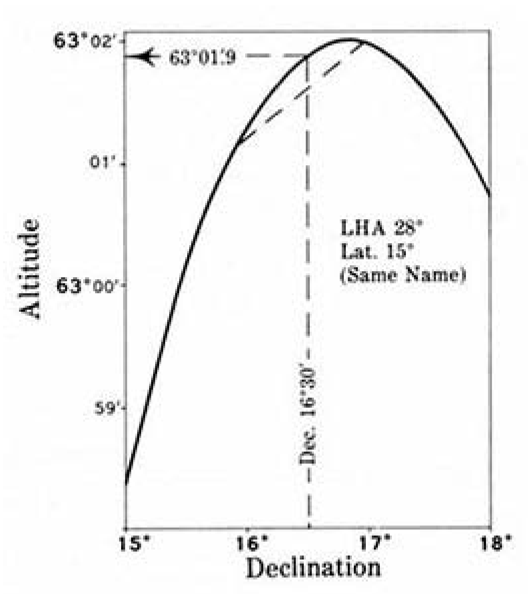


Figure 205d. Graph of Table 205c Data.

algebraically, the tabular altitude difference immediately above the respondent altitude difference from the tabular altitude difference immediately below. The result will always be a negative value.

The Double-Second Difference Correction. As shown in Figure 205a, that compartment of the DSD table opposite the block in which the Dec. Inc. is found is entered with the DSD to obtain the DSD correction to the altitude. The correction is always plus. Therefore, the sign of the DSD need not be recorded. When the DSD entry corresponds to an exact tabular value, always use the upper of the two possible corrections.

Example of the Use of the Double-Second Difference. As an example of the use of the double-second difference (DSD) the computed altitude and true azimuth are determined for Lat. 15°N, LHA 28°, and Dec. 16°30.0'N. Data are exhibited in Figure 205a.

The respondents for the entering arguments (Lat. 15° Same Name as Declination, LHA 28°, and Dec. 16°) are:

tabular altitude,	ht	63°01.2'
altitude difference,	d	(+)0.8'
azimuth angle,	Z	84.1°

Table 205e.

The linear interpolation correction to the tabular altitude for Dec. Inc. 30.0' is (+)0.4'.

$$\text{Hc} = \text{ht} + \text{linear correction} + \text{DSD correction}$$

However, by inspection of Figure 205d, illustrating this solution graphically, the computed altitude should be 63°01.9'. The actual change in altitude with an increase in declination is nonlinear. The altitude value lies on the curve between the points for declination 16° and declination 17° instead of the straight line connecting these points.

The DSD is formed by subtracting, algebraically, the tabular altitude difference immediately above the respondent altitude difference from the tabular altitude difference immediately below. Thus, the DSD is formed by algebraically subtracting (+)2.8' from (-)1.3'; the result is (-)4.1'.

As shown in Figure 205f, that compartment of the DSD table opposite the block in which the Dec. Inc. (30.0') is found is entered with the DSD (4.1') to obtain the DSD correction to the altitude. The correction is 0.3'. The correction is always plus.

$$\text{Hc} = \text{ht} + \text{linear correction} + \text{DSD correction}$$
$$\text{Hc} = 63°01.2' + 0.4' + 0.3' = 63°01.9'$$

Extrapolation.-The extending of a table is usually performed by assuming that the difference between the last few tabulated entries will continue at the same rate. This assumption is strictly correct only if the change is truly linear, but in most tables the assumption provides satisfactory results for a *slight* extension beyond tabulated values. The extent to which the assumption can be used reliably can often be determined by noting the last few differences. If the "second differences" (differences between consecutive differences) are nearly zero, the curve is nearly a straight line, for a short distance. But if consecutive second differences are appreciable, extrapolation is not reliable. For examples of linear and nonlinear relationships, refer to the first page of Table 3 (Common Logarithms of Trigonometric Functions) and compare the tabulated differences of the logarithms of secant (approximately linear on this page) and sine (nonlinear on this page).

As an example of extrapolation, consider Table 22 (Amplitudes). Suppose the amplitude for latitude 45°, declination 24.3° is desired. The last declination entry is 24.0°. The amplitude for declination 23.5° is 34.3°, and for declination 24.0° it is 35.1°. The difference is (+) 0.8°. Assuming this same difference between declinations 24.0° and 24.5°, one finds the value for 24.3° is $35.1° + \left(\frac{3}{5} \quad 0.8°\right) = 35.6°$. Below latitude 50° this table is so nearly linear that extrapolation can be carried to declination 30° without serious error.

For double or triple extrapolation, differences are found as in single interpolation.

206. General Comments

As a general rule, the final answer should not be given to greater precision than tabulated values. A notable exception to this rule is the case where

28°, 332° L.H.A.

LATITUDE SAME NAME

Dec.	15° Hc	15° d	15° Z	16° Hc	16° d	16° Z	17° Hc	17° d	17° Z
°	° ′	′	°	° ′	′	°	° ′	′	°
9	61 59.3	+14.7	99.1	61 48.8	+16.8	101.0	61 36.4	+18.9	102.8
10	62 14.0	+12.8	97.1	62 05.6	+15.0	98.9	61 55.3	+17.1	100.8
11	62 26.8	+10.9	95.0	62 20.6	+13.1	96.9	62 12.4	+15.3	98.8
12	62 37.7	+8.9•	92.8	62 33.7	+11.1	94.8	62 27.7	+13.4	96.7
13	62 46.6	+6.9•	90.7	62 44.8	+9.2	92.6	62 41.1	+11.4	94.6
14	62 53.5	+4.9•	88.5	62 54.0	+7.2•	90.5	62 52.5	+9.5•	92.4
15	62 58.4	+2.8•	86.3	63 01.2	+5.1•	88.3	63 02.0	+7.4•	90.2
16	63 01.2	+0.8•	84.1	63 06.3	+3.1•	86.1	63 09.4	+5.4•	88.0
17	63 02.0	−1.3•	81.9	63 09.4	+1.0•	83.9	63 14.8	+3.3•	85.8
18	63 00.7	−3.3•	79.7	63 10.4	−1.0•	81.6	63 18.1	+1.3•	83.6
19	62 57.4	−5.4•	77.5	63 09.4	−3.1•	79.4	63 19.4	−0.9•	81.4

Data from Page 58 of Pub 229 - Volume 2

INTERPOLATION TABLE

Dec. Inc.	Altitude Difference (d) Tens 10′	20′	30′	40′	50′	Decimals	Units 0′	1′	2′	3′	4′	5′	6′	7′	8′	9′
′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′	′
30.0	5.0	10.0	15.0	20.0	25.0	.0	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.6	4.1	4.6
30.1	5.0	10.0	15.0	20.0	25.1	.1	0.1	0.6	1.1	1.6	2.1	2.6	3.1	3.6	4.1	4.6
30.2	5.0	10.0	15.1	20.1	25.1	.2	0.1	0.6	1.1	1.6	2.1	2.6	3.2	3.7	4.2	4.7
30.3	5.0	10.1	15.1	20.2	25.2	.3	0.2	0.7	1.2	1.7	2.2	2.7	3.2	3.7	4.2	4.7
30.4	5.1	10.1	15.2	20.3	25.3	.4	0.2	0.7	1.2	1.7	2.2	2.7	3.3	3.8	4.3	4.8
30.5	5.1	10.2	15.3	20.3	25.4	.5	0.3	0.8	1.3	1.8	2.3	2.8	3.3	3.8	4.3	4.8
30.6	5.1	10.2	15.3	20.4	25.5	.6	0.3	0.8	1.3	1.8	2.3	2.8	3.4	3.9	4.4	4.9
30.7	5.1	10.3	15.4	20.5	25.6	.7	0.4	0.9	1.4	1.9	2.4	2.9	3.4	3.9	4.4	4.9
30.8	5.2	10.3	15.4	20.6	25.7	.8	0.4	0.9	1.4	1.9	2.4	2.9	3.5	4.0	4.5	5.0
30.9	5.2	10.3	15.5	20.6	25.8	.9	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0

Double Second Diff.	and Corr.
′	′
0.8	
	0.1
2.4	
	0.2
4.0	
	0.3
5.6	
	0.4
7.2	
	0.5
8.8	
	0.6
10.4	
	0.7
12.0	
	0.8
13.6	
	0.9
15.2	
	1.0
16.8	

Data from Interpolation Table

Figure 205f. Interpolation blocks from Pub No. 229.

tabulated values are known to be exact, as in Table 201a. A slight increase in accuracy can sometimes be attained by retaining one additional place in the solution until the final answer. Suppose, for instance, that the corrections for triple interpolation are (+)0.2, (+)0.3, and (-)0.3. The total correction is (+)0.2. If the total correction, rounded to tenths, had been obtained from the sum of (+)0.17, (+)0.26, and (-)0.34, the correct total would have been (+)0.09 = (+)0.1. The retaining of one additional place may be critical if the correction factors end in 0.5. Thus, in double interpolation, one correction value might be say (+)0.15, and the other one (-)0.25. The correct total is (-)0.1. But if the individual differences are rounded to (+)0.2 and (-)0.2, the total is 0.0.

The difference used for establishing the proportion is also a matter subject to some judgment. Thus, if the latitude is 17°14.6', it might be rounded to 17.2° for many purposes. Slightly more accurate results can sometimes be obtained by retaining the minutes, using $\frac{14.6}{60}$ instead of 0.2. If the difference to be multiplied by this proportion is small, the increase in accuracy gained by using the more exact value is small, but if the difference is large, the gain might be considerable. Thus, if the difference is 0.2°, the correction by using either $\frac{14.6}{60}$ or 0.2 is less than 0.05°, or 0.0° to the nearest 0.1°. But if the difference is 3.2°, the value by $\frac{14.6}{60}$ is 0.8°, and the value by 0.2 is 0.6°.

If the tabulated entries involved in an interpolation are all positive or all negative, the interpolation can be carried out on either a numerical or an algebraic basis. Most navigators prefer the former, carrying out the interpolation as if all entries were positive, and giving to the interpolated value the common sign of all entries. When both positive and negative entries are involved, all differences and corrections should be on an algebraic basis, and careful attention should be given to signs. Thus, if single interpolation is to be performed between values of (+)0.9 and (-)0.4, the difference is $0.9 - (-0.4) = 0.9 + 0.4 = 1.3$ 0. If the correction is 0.2 of this difference, it is (-)0.3 if applied to (+)0.9, and (+)0.3 if applied to (-)0.4. In the first case, the interpolated value is $(+)0.9 - 0.3 = (+)0.6$. In the second case, it is $(-)0.4 + 0.3 = (-)0.1$. If the correction had been 0.4 of the difference, it would have been (-)0.5 in the first case, and (+)0.5 in the second. The interpolated value would have been $(+)0.9 - 0.5 = (+)0.4$, or $(-)0.4 + 0.5 = (+)0.1$, respectively.

Because of the variety in methods of interpolation used, solutions by different persons may differ slightly.

CHAPTER 3

NAVIGATIONAL ERRORS

DEFINING NAVIGATIONAL ERRORS

300. Introduction

As commonly practiced, navigation is not an exact science. A number of approximations which would be unacceptable in careful scientific work are used by the navigator, because greater accuracy may not be consistent with the requirements or time available, or because there is no alternative.

Thus, when the navigator uses his latitude graduations as a mile scale or computes a great-circle course and distance, s/he neglects the flattening of the earth at the poles, a practice that is not acceptable to the geodetic surveyor. When the navigator plots a visual bearing or an azimuth line for a celestial line of position, s/he uses a rhumb line to represent a great circle on a Mercator chart. When s/he plots the celestial line of position, s/he substitutes a rhumb line for a small circle. When the navigator interpolates in sight reduction or lattice tables, s/he assumes a linear (constant-rate) change between tabulated values. When s/he measures distance by radar or depth by echo sounder, s/he assumes that the radio- or sound-wave has constant speed under all conditions. When the navigator applies dip and refraction corrections to his or her sextant altitude, s/he generally assumes standard atmospheric conditions. These are only a few of the approximations commonly applied by a navigator.

There are so many that there is a natural tendency for some of them to cancel others. Thus, under favorable conditions, a position at sea determined from celestial observation by an experienced observer should seldom be in error by more than 2 miles. However, if the various small errors in a particular observation all have the same sign (all plus or all minus), the error might be several times this amount without any mistake having been made by the navigator.

Greater accuracy could be attained, but at a price. The navigator is a practical individual. In the course of ordinary navigation, s/he would rather spend 10 minutes determining a position having a probable error of plus or minus 2 miles, than to spend several hours learning where s/he was to an accuracy of a few meters. But if the navigator can determine a recent or present position to greater accuracy, the decrease in error is attractive. The various navigational aids have been designed with this in mind. Greater accuracy in plotting could be achieved by increasing the scale of the chart or plotting sheet. This has been done for confined waters where a higher degree of accuracy is needed, but a large scale plotting sheet would be a nuisance at sea. The hand-held marine sextant is not sufficiently accurate for use in determining an astronomical position in a geodetic survey. But, it is much more satisfactory at sea than the surveyor's astrolabe or theodolite, which require stable platforms if their potential accuracy is to be realized.

An understanding of the kinds of errors involved in navigation, and of the elementary principles of probability, should be of assistance to a navigator in interpreting his or her results.

301. Definitions

The following definitions apply to the discussions of this chapter:

Error is the difference between a specific value and the correct or standard value. As used here it does not include mistakes, but is related to lack of perfection. Thus, an altitude determined by marine sextant is corrected for a standard amount of refraction, but if the actual refraction at the time of observation varies from the standard, the value taken from the table is in error by the difference between standard and actual refraction. This error will be compounded with others in the observed altitude. Similarly, depth determined by echo sounder is in error, among other things, by the difference between the actual speed of sound waves in the water and the speed used for calibration of the instrument. The depth will also be in error if an echo is returned from a phantom bottom instead of from the actual bottom. This chapter is concerned primarily with the deviation from standards. Thus, while variation of the compass is an error when referred to true directions, the difference between the assumed variation and that actually existing is an error with reference to magnetic direction. Corrections can be applied for standard values of error. It is the deviation from standard, as well as mistakes, that produce inaccurate results in navigation. Various kinds of errors are discussed in the following articles.

Mistake is a blunder, such as an incorrect reading of an instrument, the taking of a wrong value from a table, or the plotting of a reciprocal bearing. The mistake is discussed in more detail in Section 312.

Standard is something established by custom, agreement, or authority as a basis for comparison. It is customary to use nautical miles for measuring distances between ports. By international agreement the nautical mile is defined as exactly 1852 meters. By authority of various countries which are parties to the agreement, this length is translated to the linear units adopted by that country. It is the fact of establishment or general acceptance that determines whether a given quantity or condition has become a standard of measure or quality.

Thus, in 1960, the standard unit of length agreed upon at the Eleventh General (International) Conference on Weights and Measures to redefine the meter was 1,650,763.73 wavelengths of the orange-red radiation in vacuum of krypton 86 corresponding to the unperturbed transition between the 2p10 and 5d5 levels. This established standard of length now serves as a basis for measurement of any physical magnitude, as the length of the meridian. Multiples and submultiples of a standard are exact. In 1959, the U.S. adopted the exact relationships of 1 yard as equal to 0.9144 meter and 1 inch as equal to 2.54 centimeters. Hence, 39.37 U.S. inches are approximately equal to 1 meter. Because 1 foot equals 12 inches by definition, and the international nautical mile has been defined as 1852 meters, the international nautical mile is equal to 6,076.11549 U.S. feet (approximately). The previous U.S. foot (6,076.10333 . feet equals 1 nautical mile) has been re-designated as the U.S. survey foot.

Frequently, a standard is chosen so that it serves as a model which approximates a mean or average condition. However, the distinction between the standard value and the actual value at any time should not be forgotten. Thus, a standard atmosphere has been established in which the temperature, pressure, density, etc., are precisely specified for each altitude. Actual conditions, however, are generally different from those defined by the standard atmosphere. Similarly, the values for dip given in the almanacs are considered standard by those who use them, but actual dip may be appreciably different from that tabulated.

Accuracy is the degree of conformance with the correct value, while **precision** is the degree of refinement of a value. Thus, an altitude determined by a marine sextant might be stated to the nearest 0.1', and yet be accurate only to the nearest 1.0' if the horizon is indistinct.

302. Systematic Errors

Systematic errors are those which follow some law by which they can be predicted. The accuracy with which a systematic error can be predicted depends upon the accuracy with which the governing law is understood. An error which can be predicted can be eliminated, or compensation can be made for it.

The simplest form of systematic error is one of unchanging magnitude and sign. This is called a **constant error**. Examples are the index error of a marine sextant, watch error, or the error resulting from a lubber's line not being accurately aligned with the longitudinal axis of the craft. In each of these cases, all readings are in error by a constant amount as long *as the adjustment remains unchanged*, and can be removed by applying a correction of equal magnitude and opposite sign. Index error and watch error can be removed by adjustment of the instrument. Lubber's line error can be removed by aligning the lubber's line with the longitudinal axis of the craft.

Another type of systematic error results from a nonstandard rate. If a watch is gaining 4 seconds per day, its readings will be in error by 1 second after an interval of 6 hours, 8 seconds at the end of 2 days, etc. This principle is used in establishing a chronometer rate (Section 1608, Volume 1, 2019 edition) for determination of chronometer error between comparisons of the chronometer with time signals. It can be eliminated by adjusting the rate. If a current is running and no allowance for it is made in the dead reckoning, the DR position is in error by an amount proportional to elapsed time. The error introduced by maintaining heading by means of an inaccurate compass is proportional to distance, as is the lateral error in a line of position plotted from an inaccurate bearing.

One of the causes of equation of time (Section 1601, Volume 1, 2019 edition) is the fact that the ecliptic, around which annual motion occurs, is not parallel to the celestial equator, around or parallel to which apparent daily motion takes place. The same type of systematic error is involved in other measurements. Consider the measurement of bearing with a tilted compass card. Bearing is measured by a system of uniform graduations (degrees) of a circle (such as a compass card) in the horizontal plane. If the card is tilted, and its graduations are projected onto the horizontal plane, the circle becomes an ellipse with the graduations unequally spaced. Along the axis of tilt and a line perpendicular to it, directions are correct. But near the axis of tilt the graduations are too close together, and near the perpendicular they are too widely spaced.

The error thus introduced is similar to that which would arise if a watch face were tilted but the motion of the hands remained horizontal. If it were tilted around the "3-9" line, it would appear to run slow near the hour and half hour, and fast near the quarter and three-quarter hours. If the direction to be observed is of an object above or below the horizontal, as the azimuth of a celestial body, measurement is made to the foot of the perpendicular through the object.

The sight vanes of a compass move in a plane perpendicular to the compass card. Hence, if the card is tilted, measurement is made to the foot of a perpendicular to the card, rather than to the foot of a perpendicular to the horizontal, introducing an error which increases with the angle of tilt and also with the angle of elevation (or depression) of the object. This error is greatest along the axis of tilt, and zero along the perpendicular to it. Both of these tilt errors can be corrected by leveling the compass card.

A different type of tilt error occurs when a reflection takes place from a tilted surface, such as the ionosphere, the error being proportional to the angle of tilt. In some respects, this error is similar to coastal refraction of a radio wave.

Additional examples of systematic error are uncorrected deviation of the compass, error due to a position in a pattern of hyperbolas, error due to incorrect location of a Loran transmitter, uncorrected parallax, and uncorrected personal error.

303. Random Errors

Random errors are chance errors, unpredictable in magnitude or sign. They are governed by the laws of probability. If the altitude of a celestial body is observed, the reading may be (1) too great, (2) correct, or (3) too small. If a number of observations are made, and there is no systematic error, the probability of a positive error is exactly equal to the probability of a negative error. This does not mean that every second observation having an error will be too great. However, the greater the number of observations, the greater is the probability that the percentage of positive errors will equal the percentage of negative ones, and that their magnitudes will correspond.

Error	No. of obs.	Percent of obs.
- 10′	0	0. 0
- 9′	1	0. 2
- 8′	2	0. 4
- 7′	4	0. 8
- 6′	9	1. 8
- 5′	17	3. 4
- 4′	28	5. 6
- 3′	40	8. 0
- 2′	53	10. 6
- 1′	63	12. 6
0	66	13. 2
+ 1′	63	12. 6
+ 2′	53	10. 6
+ 3′	40	8. 0
+ 4′	28	5. 6
+ 5′	17	3. 4
+ 6′	9	1. 8
+ 7′	4	0. 8
+ 8′	2	0. 4
+ 9′	1	0. 2
+10′	0	0. 0
0	500	100. 0

Table 303. Normal distribution of random errors.

Suppose that 500 observations are made, with the results shown in Table 303. A close approximation of the plot of these errors is shown in Figure 303a. The plot has been modified slightly to constitute the normal curve of random errors, which is the same as the actual curve except that the normal curve approaches zero as the error increases, while the actual curve reaches zero at (+)10' and (-)10'. The height of the curve at any point represents the percentage of obser-

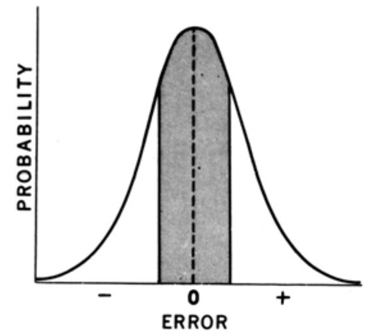


Figure 303a. Normal curve of random error with 50 percent of area shaded. Limits of shaded area indicate probable error.

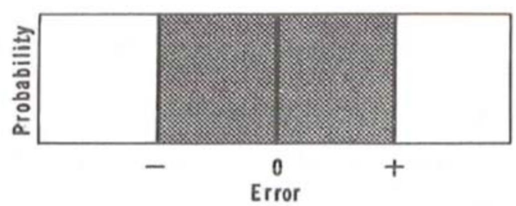


Figure 303b. Rectangular error, with 50 percent area shaded.

vations that can be expected to have the error indicated at that point. The probability of any similar observation having any given error is the proportion of the number of observations having this error to the total number of observations, or the percentage expressed as a decimal. Thus, the probability of an observation having an error of -3' is

$$\frac{40}{500} = \frac{1}{12.5} = 0.08(8\%)$$

If the area under the curve represents 100 percent of the observations, half the area (the shaded portion of Figure 303c) represents 50 percent of the observations. The value of the error at the limits of this shaded portion is often called the “50 percent error,” or **probable error**, meaning that 50 percent of the observations can be expected to have less error, and 50 percent greater error. Similarly, the limits which contain the central 95 percent of the area denote the 95 percent error. The percentage of error is found mathematically. For a normal curve, each error is squared, the sum of the squares is divided by one less than the number of observations, and the square root of the quotient is determined. This value is called the **standard deviation** or **standard error** (σ, the Greek letter sigma). In the illustration, the standard deviation is the square root of:

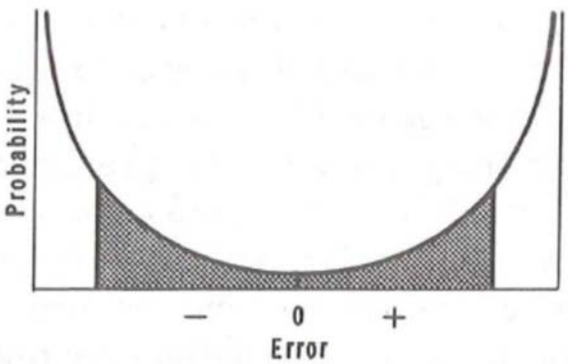


Figure 303c. Periodic error, with 50 percent area shaded.

$$0 \quad (-10)^2 + 1 \quad (-9)^2 + 2 \quad (-8)^2 + 4 \quad (-7)^2 + 9 \quad (-6)^2, \text{etc}$$

divided by 499 or

$$\frac{4474}{\sqrt{499}} = \sqrt{8.966} = 2.99 \text{ (about 3)}$$

The standard deviation is the 68.27 percent error. The probability of the occurrence of an error of or less than a specific magnitude may be approximately determined by the following relationship (with the answers for the illustration given):

50% error = 2/3 x σ = 2' (approx.)

68% error = 1 x σ = 3' (approx.)

95% error = 2 x σ = 6' (approx.)

99% error = 2 2/3 x σ = 8' (approx.)

99.9% error =3 1/3 x σ = 10' (approx.)

Many of the errors of navigation do not follow the normal distribution discussed above. *Pub. No. 229* values of altitude can be taken only to the nearest 0.1'. The error in tabular altitude might have any value from (+) 0. 05' to (-) 0.05', and any value within these limits is as likely to occur as any other of the same precision. The same is true of a sextant that cannot be read more precisely than 0.1', and of a time-difference that cannot be measured more precisely than 1 µs. These values refer to the single errors indicated, and not to the total error that might be involved. This is a **rectangular error**, so called because of the shape of its plot, as shown in Figure 303b. The 100 percent error is half the difference between readings. The 50 percent error is half this amount, the 95 percent error is 0.95 times this amount, etc. In some cases it may be more meaningful to refer to the rectangular error as the **resolution error**.

Still another type random error is encountered in navigation. If a compass is fluctuating periodically due to yaw of a ship, its motion slows as the end of a swing is approached, when the error approaches maximum value. If readings were taken continuously or at equal intervals of time, the interval being a small percentage of the total period of oscillation, the curve of errors would have a characteristic U-shape, as shown in Figure 303c. The same type error is involved in measurement of altitude of a celestial body from a wing of the bridge of a heavily rolling vessel, when the roll causes large changes in the height of eye. This type of error is called a **periodic error**. The effect is accentuated by the tendency of the observer to make readings near one of the extreme values because the instrument appears steadiest at this time. If it is impractical to make a reading at the center of the period, the error can be eliminated or reduced by averaging readings taken continuously or at short intervals, as indicated above. This is the method used in averaging type artificial-horizon sextants. Generally, better results can be obtained by taking maximum positive and maximum negative readings, and averaging the results.

The curve of any type of random error is symmetrical about the line representing zero error. This means that in the ideal plot every point on one side of the curve is error of the same magnitude. The average of all readings, considering signs, is zero. The larger the number of readings made, the greater the probability of the errors fitting the ideal curve. Another way of stating this is that as the number of readings increases, the error of the average can be expected to decrease

304. Combinations of Errors

Many of the results obtained in navigation are subject to more than one error. Chapter 19, Volume 1, lists 19 errors applicable to sextant altitudes. Some of these have several components. A number of possible errors are involved in the determination of computed altitude and azimuth. A rectangular error is possible in finding the altitude difference. Several additional errors may affect the accuracy of plotting. Thus, the line of position as finally plotted may include 30 errors or more. Corrections are applied for some of the larger ones, so that in each of these cases the applicable error is the difference between the applied correction and the actual error. Thus, a dip correction may be applied for a height of eye of 30 feet, while the actual height at the moment of observation may be 31 feet 6 inches. Even if the height of eye is exactly 30 feet, a rectangular error may be involved in taking the dip correction from the table

If two or more errors are applicable to a given result, the total error is equal to the algebraic sums of all errors. Thus, if a given number is subject to errors of (+) 4, (-) 2, (-) 1, (+) 3, (+) 2, 0, and (-) 2, the total error is (+) 4. Systematic errors can be combined by adding the curves of

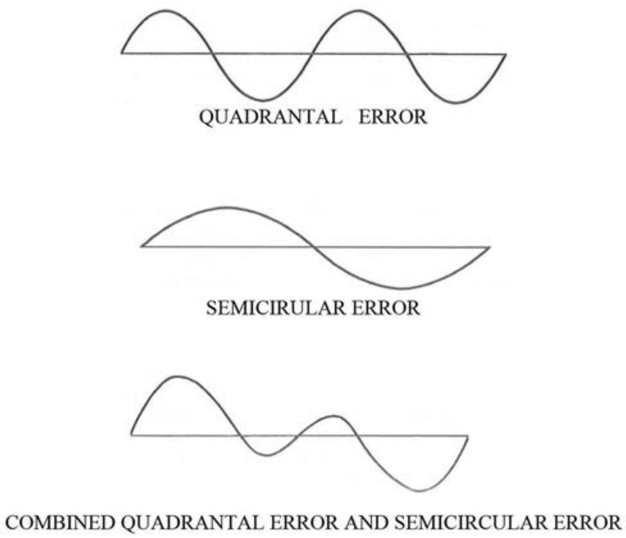


Figure 304. Combining systemic error.

individual errors. Thus, a magnetic compass may have a quadrantal error as shown by the top curve of Figure 304, and a semicircular error as shown by the second curve. The sum of these two errors is shown in the bottom curve. If, in addition, the compass has a constant error, the bottom curve is moved vertically upward or downward by the amount of the constant error, without undergoing a change of form. If the constant error is greater than the maximum value of the combined curves, all errors are positive or all are negative, but of varying magnitude.

If a number of random errors are combined, the result tends to follow a normal curve regardless of the shape of the individual errors, and the greater the number, the more nearly the result can be expected to approach the normal curve (Figure 303a). If a given result is subject to errors of plus or minus 3, 2, 1, 2, 4, 2, 1, 8, 1, and 2, the total error could be as much as 26 if all errors had the same sign. However, if these are truly random, the probability of them all having the same sign is only 1 in 1024. This is so because the chance of any one being positive (or negative) is one half. By the same reasoning, approximately half of the positive (or negative) results will have any one particular additional correction positive (or negative). Thus, the probability of any two particular corrections having a positive (or negative) sign is $1/2 \quad 1/2 = (1/2)^2 = \frac{1}{4}$. The probability of all 10 corrections having a positive (or negative) sign is $(1/2)^{10} = \frac{1}{1024}$. If there were 20 corrections, the probability of all having a positive (or negative) sign would be $(1/2)^{20} = \frac{1}{1048576}$.

When both systematic and random errors are present in a process, both effects are present. An increase in the number of readings decreases the residual random error, but regardless of the number of readings, a systematic error is present in its entirety. Thus, if a number of phase-difference readings are made at a fixed point, the average should be a good approximation of the true value if there is no systematic error. But if the equipment is out of adjustment to the extent that the lane is incorrectly identified, no number of readings will correct this error. In this illustration, a constant error is combined with a normal random error. The normal curve has the correct shape, but is offset from the zero value.

Under some conditions, systematic errors can be eliminated from the results even when the magnitude is not determined. Thus, if two celestial bodies differ in azimuth by 180°, and the altitude of each is observed, the line midway between the lines of position resulting from these observations is free from any *constant* error in the *altitude* (such as abnormal refraction or dip, or incorrect IC). It would *not* be free from such a constant error as one in time (unless the bodies were on the celestial meridian). Similarly, a fix obtained by observations of three stars differing in azimuth by 120°, or four stars differing by 90° is free from constant error in the altitude, if the center of the figure made by the lines of position is used. The center of the figure formed by circles of position from distances of objects equally spaced in azimuth is free from a constant error in range. A constant error in bearing lines does not introduce an error in the fix if the objects are equally spaced in azimuth. In all of these examples, the correct position is *outside* the figure formed by the lines of position if all objects observed are on the same side of the observer (that is, if they lie within an arc of less than 180°).

305. Navigation Accuracy

Navigation accuracy is normally expressed in terms of the probability of being within a specified distance of a desired point during the navigation process.

If the accuracy of only a single line of position is being considered, the specified distance may be stated as the standard deviation (Section 303) or some multiple thereof, assuming that the errors of the line of position follow a **single-axis normal distribution.** The distance as stated for the standard deviation of a line of position is measured from the arithmetic mean of the positions which could be established from a large number of observations at a given place and time. Therefore, this distance does not indicate the separation between the line of position and the observer's actual position, except by chance. If the error is stated as 1σ, 68.27 percent of the cases should result in line of position displacements from the arithmetic mean in any direction not exceeding the distance specified for 1σ. If the error is stated as 2σ, 95.45 percent of the lines of position should not be displaced from the arithmetic mean in any direction by more than the distance specified for 2σ. If the error is stated as the probable error, 50 percent of the lines

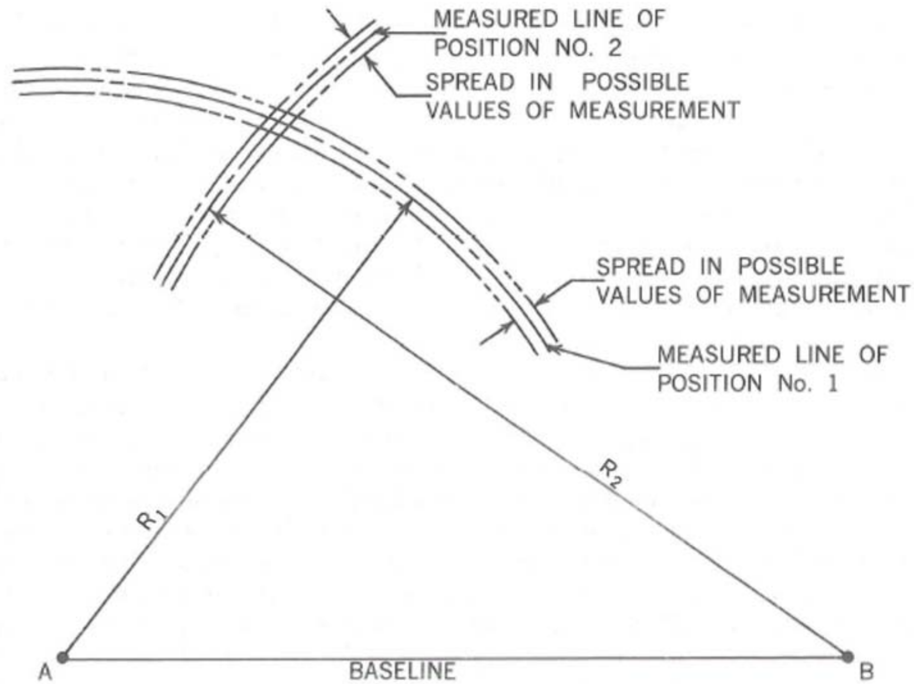


Figure 305a. Fix established at intersection of two lines of position having different values of error.

of position should not be displaced from the arithmetic mean in any direction by more than the distance specified for 0.6745σ.

The standard deviation is also employed in developing expressions for the probability of a fix position being within a specified distance of the mean of the positions which could be established from a large number of observations at a given place and time by means of the system used to establish the fix.

In the following discussion, the fix is established by the intersection of two lines of position, each of which may be in error. The lines of position (Figure 305a) are range measurements from two points at the extremities of a baseline of known length. Because of inaccuracies in measurement, the actual ranges differ from the measured values and may lie somewhere between the limits which are shown as additional arcs either side of the measured arc.

The intersection of the two lines of position together with the standard deviations associated with each is drawn to an expanded scale in Figure 305b. It can be shown that the *contours of equal probability density* about such an intersection are ellipses with their center at the intersection. Thus, the ellipse shown in Figure 305b might be the 75 percent probability ellipse, meaning that there are three chances in four that a fix will lie within such ellipse centered upon the mean of the positions which would be established from a large number of observations at a given place and time by means of the system used to establish the fix.

For simplicity in this discussion of navigation accuracy, the following assumptions are made:

1. All constant errors or **bias errors** have been removed, leaving only the random errors. Thus, the mean or average error is assumed to be zero.
2. These random errors are assumed to be normally distributed.
3. The errors associated with the two intersecting lines of position are assumed to be independent. This assumption implies that a change in the error of one line of position has no effect upon the other.
4. The lines of position are assumed to be straight lines in the small area in the immediate vicinity of their intersection. This assumption is valid so long as the standard deviation is small compared to the radius of curvature of the line of position.
5. Errors of position are limited to the two-dimensional case. As shown in Figure 305b, the general case of the intersection of two lines of position at any angle of cut and with different values of error associated with each line of position results in an elliptical error figure. Figure 305c shows the ellipse simplified to geometrical terms.

One may readily surmise from Figure 305c that the exact shape of the error figure varies with the magnitudes of the two one-dimensional input errors, $\sigma 1$ and $\sigma 2$ as well as with the angle of cut, α. The angle α is also the angle between the two values of sigma because the standard

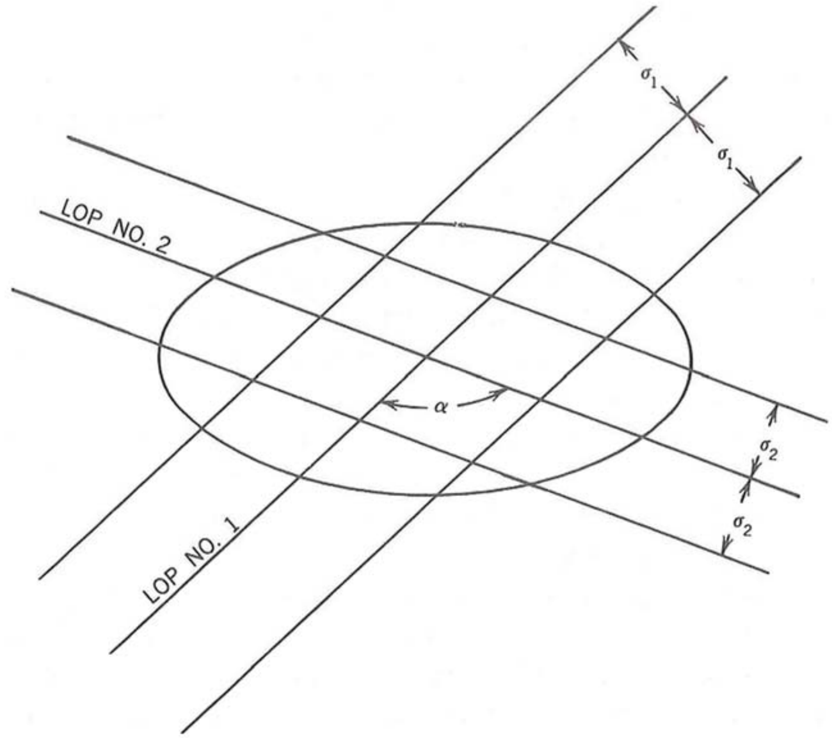


Figure 305b. Expanded view of intersection of two lines of position.

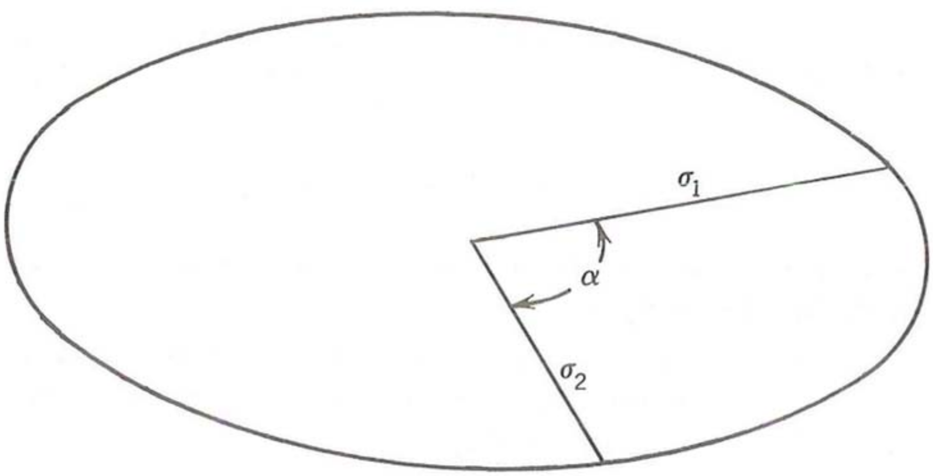


Figure 305c. Basic error ellipse.

deviations are mutually perpendicular to their corresponding lines of position. These variations can be calculated to provide the probability that a point is located within a circle of stated radius.

When this is done, the error is stated in terms more meaningful to the practicing navigator. The basis of this concept may best be seen by first considering the special case when the two errors are equal, and the angle of intersection of the lines of position is a right angle. In this case, *and in this case alone*, the error figure becomes a circle and is described by the circular normal distribution. A plot of this special function is given in Figure 305d. In this plot, the horizontal axis is measured in terms of R/σ, R being the stated radius of the circle and σ being the measure of error. The error measure is given simply as σ, for in this circular case $\sigma 1 = \sigma 2$. To illustrate, a measurement system gives a circular error figure and has a value of $\sigma = 100$ meters; the probability of actually being located within a circle of 100 meters radius when $R/\sigma = 1.0$ may be read from the verti-

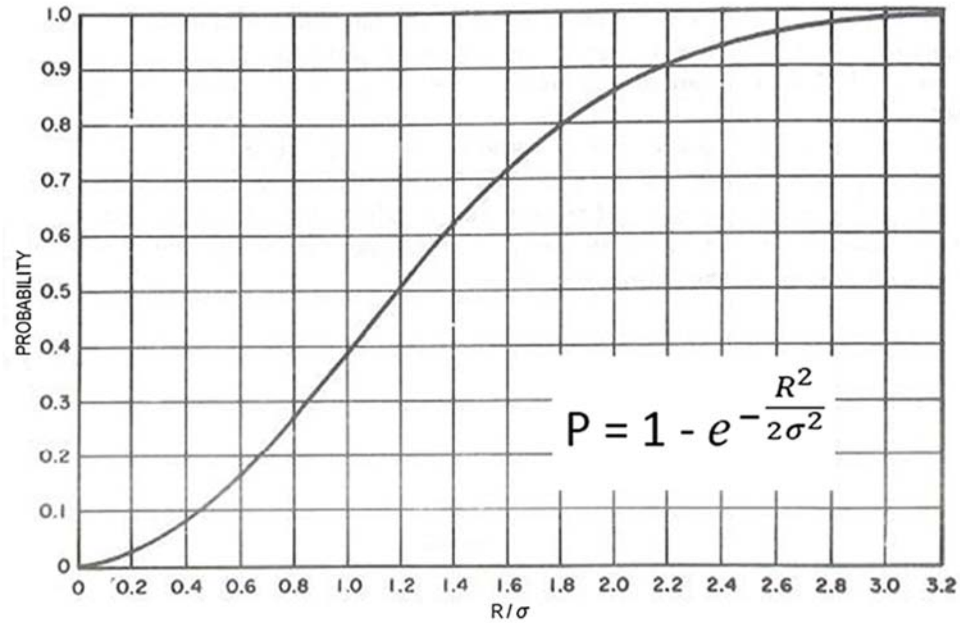


Figure 305d. Circular normal distribution.

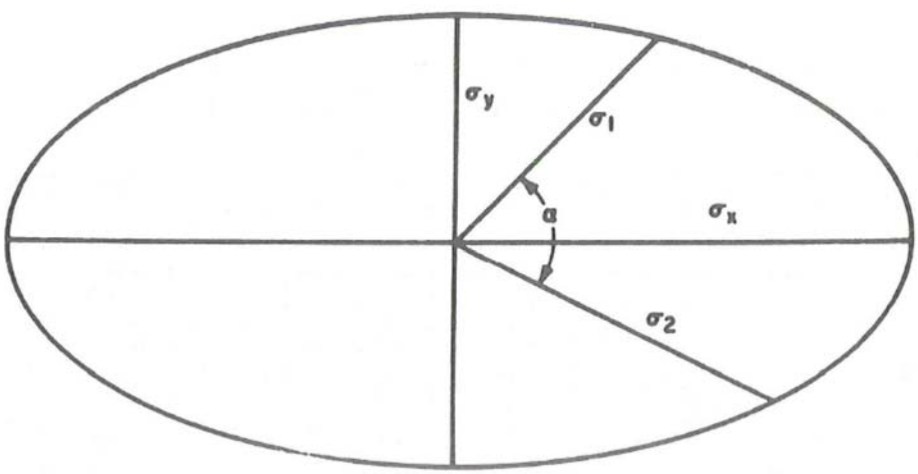


Figure 305e. Transformation to standard deviations along ellipse axes.

cal axis to be 39.3 percent. To obtain the radius of a circle within which a 50 percent probability results, the corresponding value of R/σ is seen to be 1.18 from the graph. Thus, for this example, the **circular probable error** (**CPE** or **CEP** or circle of 50% probability) would be 118 meters..

In one method of using error ellipses to obtain the radii of **circles of equivalent probability**, new values of σ are found along the major and minor axes of the ellipse (Figure 305e) using the following equations:

$$\sigma x^2 = \frac{1}{2\sin^2\alpha}\left[\sigma 1^2 + \sigma 2^2 + \sqrt{(\sigma 1^2 + \sigma 2^2)^2 - 4\sin^2\alpha\,\sigma_1^2\sigma_2^2}\right]$$

$$\sigma y^2 = \frac{1}{2\sin^2\alpha}\left[\sigma 1^2 + \sigma 2^2 - \sqrt{(\sigma 1^2 + \sigma 2^2)^2 - 4\sin^2\alpha\,\sigma_1^2\sigma_2^2}\right].$$

Then the ratio $c = \frac{\sigma_y}{\sigma_z}$ where σ_x is the larger of the two new standard deviations, is used in entering Table 305a which relates ellipses of varying values of ellipticity to the radii of circles of equivalent probability.

For a numerical example to illustrate the method of calculation, assume that the angle of cut α is 50°, σ 1 is 15 meters, and σ 2 is 20 meters to determine the probability of location within a circle of 30 meters radius.

For the computation the following numbers are needed:

$$\sigma_1^{\ 2} = 225$$

$$\sigma_2^{\ 2} = 400$$

$$\sin 2\sigma = 0.5868$$

Substituting in the equations for $\sigma_x^{\ 2}$ and $\sigma_y^{\ 2}$, σ_x and σ_y are calculated as 29.9 meters and 13.1 meters, respectively. Since the function *K* multiplied by the larger of the two standard deviations obtained by the transformation method gives the value of the radius of the circle of the corresponding value of probability shown in Table 305a, *K*=1.003. On entering Table 305a with *K*=1.0 and c= 0.44, the probability is found to be 62 percent.

Table 305b and Figure 305g provide ready information about the sizes of circles of specific probability value associated with ellipses of varying eccentricities.

In another method, fictitious values of sigma of identical value, indicated by σ *, are assumed to replace the two unequal values originally given (σ 1and σ 2). A fictitious angle of cut α * is also assumed to replace the angle of cut (α) originally given (Figure 305f).

The method utilizes a set of probability curves, with a separate curve for each value of angle of cut (Figure 305h). These curves can be used only when the two error measures are equal, hence the need for making the transformation to the fictitious σ *.

The values of σ * and α * needed to utilize the probability curves may either be determined from Figure 305j and Figure 305i or by means of the following equations:

$$\sigma^* = \frac{\sin\beta\sqrt{\sigma_1^{\ 2} + \sigma_2^{\ 2}}}{\sqrt{2}}$$

$$\alpha^* = arc\sin(\sin 2\beta \sin\alpha)$$

where

$$\beta = arc\tan(\sigma_1/\sigma_2)$$

Thus,

$$\sin 2\beta = \frac{2\sigma_1\sigma_2}{\sigma_1^{\ 2} + \sigma_2^{\ 2}}.$$

To use the curve and nomogram for obtaining σ * and α *, one must first calculate the ratio σ_2/σ_1. The value σ_1, is always taken as the larger of the two in the ratio so that the ratio is always less than 1.0. With this ratio, enter the curve of Figure 305j and obtain the σ *factor. Multiply σ_1 by this factor to obtain the fictitious function σ *. The nomogram of Figure 305i is entered with the same ratio to obtain the fictitious angle of cut α *.

For a numerical example to illustrate the method of calculation, assume that the angle of cut of 50°, σ_1, is 20 meters, and σ_2 is 15 meters to determine the probability of location within a circle of 30 meters radius.

Calculate the ratio $\sigma_2/\sigma_1 = \frac{15}{20} = 0.75$.

Enter the curve of Figure 305j with this ratio and obtain the σ * factor (0.845). Multiply this factor by σ_1 to obtain σ * equals 16.9 meters. Calculate the ratio

$$R/\sigma^* = 30/16.9 = 1.78.$$

Enter the nomogram of Figure 305i with the ratio σ_2/σ_1, and with the given angle α to obtain the fictitious angle of cut α * = 47°.

The values $R/\sigma^* = 1.78$ and $\alpha^* = 47°$ are then used to enter the probability curves of to obtain P= 0.62 or 62 percent, interpolating between the 40° and 50° curves for $\alpha^* = 47°$.

GEOMETRIC ERROR CONSIDERATIONS

306. Geometric Error Considerations

From the information that can be derived using the two methods of transformation of elliptical error data, one can develop curves which show for constant values of initial error that the size of a circle of fixed value of probability varies as a function of the angle of cut of the lines of position.

To simplify the investigation of geometrical factors, it is initially desirable to consider the special case of $\sigma_1 = \sigma_2 = \sigma$. Under this special condition, the long equations for σ_x and σ_y can be simplified to facilitate computation as follows:

$$\sigma_x = \frac{\sqrt{2}}{2\sin\frac{1}{2}\alpha}\sigma \qquad (\sigma_1 = \sigma_2)$$

$$\sigma_y = \frac{\sqrt{2}}{2\cos\frac{1}{2}\alpha}\sigma \qquad (\sigma_1 = \sigma_2)$$

Taking the ratio of these two values, a simple equation is found for the ratio *c*

$$c = \frac{\sigma_y}{\sigma_x} = \tan\frac{1}{2}\alpha$$

K \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.0796557	.0443987	.0242119	.0164176	.0123875	.0099377	.0082940	.0071157	.0062299	.0055400	.0049875
0.2	.1585194	.1339783	.0884533	.0628396	.0482413	.0390193	.0327123	.0281415	.0246824	.0219757	.0198013
0.3	.2358228	.2213804	.1739300	.1318281	.1039193	.0851535	.0719102	.0621386	.0546598	.0487639	.0440025
0.4	.3108435	.3010228	.2635181	.2139084	.1742045	.1451808	.1237982	.1076237	.0950495	.0850326	.0768837
0.5	.3829249	.3755884	.3481790	.3003001	.2532953	.2152886	.1857448	.1626829	.1443941	.1296286	.1175031
0.6	.4514938	.4457708	.4255605	.3846374	.3357384	.2914682	.2548177	.2251114	.2009797	.1811783	.1647298
0.7	.5160727	.5115048	.4960683	.4633258	.4170862	.3699305	.3280302	.2925654	.2629373	.2381583	.2172955
0.8	.5762892	.5725957	.5604457	.5349387	.4941882	.4474207	.4025628	.3627122	.3283453	.2989700	.2738510
0.9	.6318797	.6288721	.6191354	.5993140	.5651564	.5213998	.4759375	.4333628	.3953279	.3620135	.3330232
1.0	.6826895	.6802325	.6723586	.6568242	.6291249	.5900953	.5461319	.5025790	.4621421	.4257553	.3934693
1.1	.7286679	.7266597	.7202682	.7079681	.6859367	.6524489	.6116316	.5687467	.5272462	.4887873	.4539256
1.2	.7698607	.7682215	.7630305	.7532175	.7359558	.7079973	.6714269	.6306168	.5893494	.5498736	.5132477
1.3	.8063990	.8050648	.8008554	.7929968	.7793550	.7567265	.7249673	.6873122	.6474394	.6079822	.5704426
1.4	.8384867	.8374049	.8340018	.8277048	.8169851	.7989288	.7720889	.7383089	.7007900	.6623035	.6216889
1.5	.8663856	.8655127	.8627728	.8577362	.8493071	.8350816	.8129287	.7833962	.7489500	.7122546	.6753475
1.6	.8904014	.8897008	.8875060	.8834914	.8768644	.8657559	.8478393	.8226246	.7917194	.7574708	.7219627
1.7	.9108691	.9103102	.9085619	.9053766	.9001746	.8915536	.8773116	.8562471	.8291137	.7977882	.7462539
1.8	.9281394	.9276964	.9263125	.9237989	.9197275	.9130680	.9019110	.8846624	.8613238	.8332175	.8021013
1.9	.9425669	.9422182	.9411299	.9391586	.9359855	.9308615	.9222277	.9083609	.8886731	.8639149	.8355255
2.0	.9544997	.9542272	.9533775	.9518415	.9493815	.9454546	.9388418	.9278799	.9115762	.8901495	.8646647
2.1	.9642712	.9640598	.9634011	.9622127	.9603170	.9573205	.9522999	.9437668	.9305013	.9122714	.8897495
2.2	.9721931	.9720304	.9715237	.9706109	.9691597	.9668845	.9631017	.9565522	.9459386	.9306821	.9110784
2.3	.9785518	.9784275	.9780408	.9773450	.9762419	.9745239	.9716934	.9667306	.9583739	.9458085	.9289946
2.4	.9836049	.9835108	.9832180	.9826918	.9818594	.9805703	.9784661	.9747495	.9682698	.9580804	.9438652
2.5	.9875807	.9875100	.9872900	.9868953	.9862720	.9853112	.9837569	.9810035	.9760522	.9679136	.9560631
2.6	.9906776	.9906249	.9904612	.9901674	.9897045	.9889934	.9878527	.9858331	.9821023	.9756969	.9659525
2.7	.9930661	.9930271	.9929062	.9926894	.9923483	.9918260	.9909944	.9895268	.9867530	.9817837	.9738786
2.8	.9948897	.9948612	.9947727	.9946141	.9943649	.9939842	.9933821	.9923249	.9902888	.9864876	.9801589
2.9	.9962684	.9962477	.9961834	.9960684	.9958878	.9956126	.9951798	.9944246	.9929482	.9900803	.9850792
3.0	.9973002	.9972853	.9972391	.9971564	.9970266	.9968294	.9965205	.9959854	.9949274	.9927925	.9888910
3.1	.9980648	.9980542	.9980212	.9979622	.9978699	.9977296	.9975109	.9971348	.9963851	.9948168	.9918113
3.2	.9986257	.9986182	.9985949	.9985533	.9984880	.9983892	.9982356	.9979733	.9974478	.9963105	.9940240
3.3	.9990332	.9990279	.9990116	.9989824	.9989368	.9988677	.9987607	.9985792	.9982147	.9974004	.9956822
3.4	.9993261	.9993225	.9993112	.9992909	.9992593	.9992115	.9991376	.9990129	.9987626	.9981868	.9969113
3.5	.9995347	.9995323	.9995245	.9995105	.9994888	.9994559	.9994053	.9993204	.9991502	.9987480	.9978125
3.6	.9996818	.9996801	.9996748	.9996653	.9996505	.9996281	.9995938	.9995364	.9994218	.9991442	.9984662
3.7	.9997844	.9997832	.9997797	.9997733	.9997633	.9997482	.9997251	.9996867	.9996102	.9994208	.9989352
3.8	.9998553	.9998545	.9998522	.9998478	.9998412	.9998311	.9998157	.9997902	.9997396	.9996119	.9992682
3.9	.9999038	.9999033	.9999018	.9998989	.9998945	.9998878	.9998776	.9998608	.9998276	.9997426	.9995020
4.0	.9999367	.9999363	.9999353	.9999334	.9999305	.9999261	.9999195	.9999085	.9998870	.9998309	.9996645
4.1	.9999587	.9999585	.9999578	.9999566	.9999547	.9999519	.9999475	.9999404	.9999266	.9998900	.9997763
4.2	.9999733	.9999732	.9999727	.9999720	.9999707	.9999689	.9999661	.9999616	.9999527	.9999292	.9998523
4.3	.9999829	.9999828	.9999826	.9999821	.9999813	.9999801	.9999783	.9999754	.9999698	.9999548	.9999034
4.4	.9999892	.9999891	.9999889	.9999886	.9999881	.9999874	.9999863	.9999845	.9999809	.9999715	.9999375
4.5	.9999932	.9999932	.9999931	.9999929	.9999925	.9999921	.9999914	.9999902	.9999881	.9999822	.9999599
4.6	.9999958	.9999957	.9999957	.9999955	.9999954	.9999951	.9999947	.9999939	.9999926	.9999889	.9999746
4.7	.9999974	.9999974	.9999973	.9999973	.9999971	.9999970	.9999967	.9999963	.9999955	.9999932	.9999840
4.8	.9999984	.9999984	.9999984	.9999983	.9999983	.9999982	.9999980	.9999977	.9999972	.9999959	.9999901
4.9	.9999990	.9999990	.9999990	.9999990	.9999990	.9999989	.9999988	.9999986	.9999983	.9999975	.9999939
5.0	.9999994	.9999994	.9999994	.9999994	.9999994	.9999993	.9999993	.9999992	.9999990	.9999985	.9999963
5.1	.9999997	.9999997	.9999997	.9999996	.9999996	.9999996	.9999996	.9999995	.9999994	.9999991	.9999978
5.2	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999997	.9999997	.9999995	.9999987
5.3	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999998	.9999997	.9999992
5.4	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999995
5.5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	.9999999	.9999999	.9999999	.9999997
5.6								1.0000000	1.0000000	.9999999	.9999998
5.7										1.0000000	.9999999
5.8											1.0000000
5.9											
6.0											

Table 305a. Circular error probability. Argument c is the ratio of the smaller standard deviation to the larger standard deviation. For the argument c and K, the table provides the probability that a point lies within a circle whose center is at the origin and whose radius is K times the larger standard deviation.

P \ c	0. 0	0. 1	0. 2	0. 3	0. 4	0. 5	0. 6	0. 7	0. 8	0. 9	1. 0
.5000	0. 67449	0. 68199	0. 70585	0. 74993	0. 80785	0. 87042	0. 93365	0. 99621	1. 05769	1. 11807	1. 17741
.7500	1. 15035	1. 15473	1. 16825	1. 19246	1. 23100	1. 28534	1. 35143	1. 42471	1. 50231	1. 58271	1. 66511
.9000	1. 64485	1. 64791	1. 65731	1. 67383	1. 69918	1. 73708	1. 79152	1. 86253	1. 94761	2. 04236	2. 14597
.9500	1. 95996	1. 96253	1. 97041	1. 98420	2. 00514	2. 03586	2. 08130	2. 14598	2. 23029	2. 33180	2. 44775
.9750	2. 24140	2. 24365	2. 25053	2. 26255	2. 28073	2. 30707	2. 34581	2. 40356	2. 48494	2. 58999	2. 71620
.9900	2. 57583	2. 57778	2. 58377	2. 59421	2. 60995	2. 63257	2. 66533	2. 71515	2. 79069	2. 89743	3. 03485
.9950	2. 80703	2. 80883	2. 81432	2. 83289	2. 83830	2. 85894	2. 88859	2. 93347	3. 00431	3. 11073	3. 25525
.9975	3. 02334	2. 02500	3. 03010	3. 03898	3. 05234	3. 07144	3. 09871	3. 13969	3. 20586	3. 31099	3. 46164
.9990	3. 29053	3. 29206	3. 29673	3. 30489	3. 31715	3. 33464	3. 35949	3. 39647	3. 45698	3. 55939	3. 71692

Table 305b. Factors for conversion of probability ellipse to circle of equivalent probability.

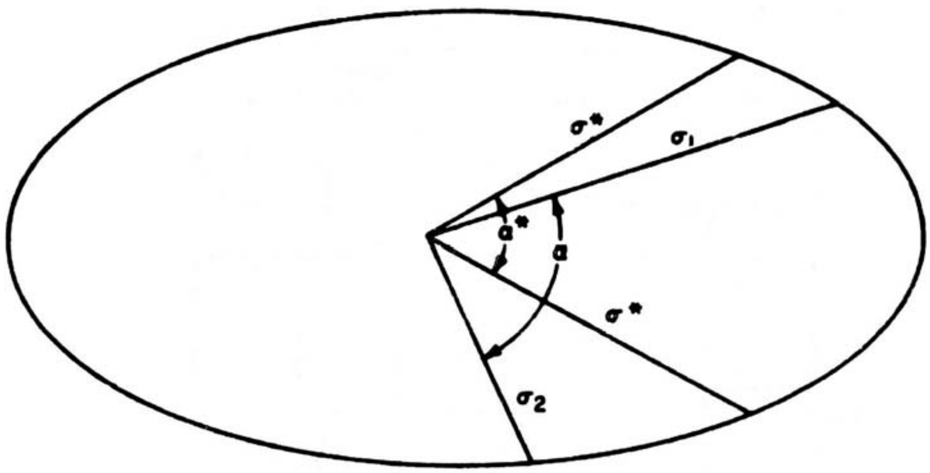


Figure 305f. Transformed parameters of error ellipse.

Utilizing these simplified equations, significant parameters of error ellipses are tabulated in Table 306a as a function of the angle of cut α . Using the CEP curve of Figure 305g, values of the CEP are calculated for each angle, showing that the CEP increases as the angle of cut decreases. The last column in the table gives the factor by which the CEP for angles less than 90° is greater than the CEP for a right angle. This magnification of error curve is plotted in Figure 306b. The curve for the 90 percent probability circle has a slightly differing shape from the CEP curve as shown in Figure 306b. Values for the 90 percent probability circle are given in table Table 306c. Figure 306b indicates the magnitude of the growth of error as the angle of cut varies from 90°.

It is also of interest to consider what values of probability result if the radius of the circle is held constant at the minimum value corresponding to that obtained for the 90° angle of cut. These values may be obtained from the probability versus angle of cut curves in .

Along the ordinate $R/\sigma = 1.177$ which corresponds to the CEP for the circular case, one may read the lesser values of probability corresponding to the various angles of cut. Likewise, one may also obtain the probability values corresponding to holding a circle the size of the 90 percent probability circle for the circular case by using the ordinate $R/\sigma = 2.15$ (also equivalent to 1.82 times the CEP). These two curves are plotted in Figure 306e and the numerical values are given in Table 306d. It is to be noted that the probability values are not inversely related to the error factors plotted in the preceding curves. The geometric error factor is a simple trigonometric function; the probability curves are exponential functions.

307. Clarification of Terminology

The following discussion is presented to insure that there is no misunderstanding with respect to the use of terms having one meaning when discussing one-dimensional errors and another when discussing two-dimensional errors.

Although the basic problem of position location is concerned with the two dimensions necessary to describe an area, one-dimensional error measures are commonly applied to each of the two dimensions involved. As demonstrated in article 305, the use of the one-dimensional standard deviation of each line of position permitted a general approach to the consideration of the error ellipse.

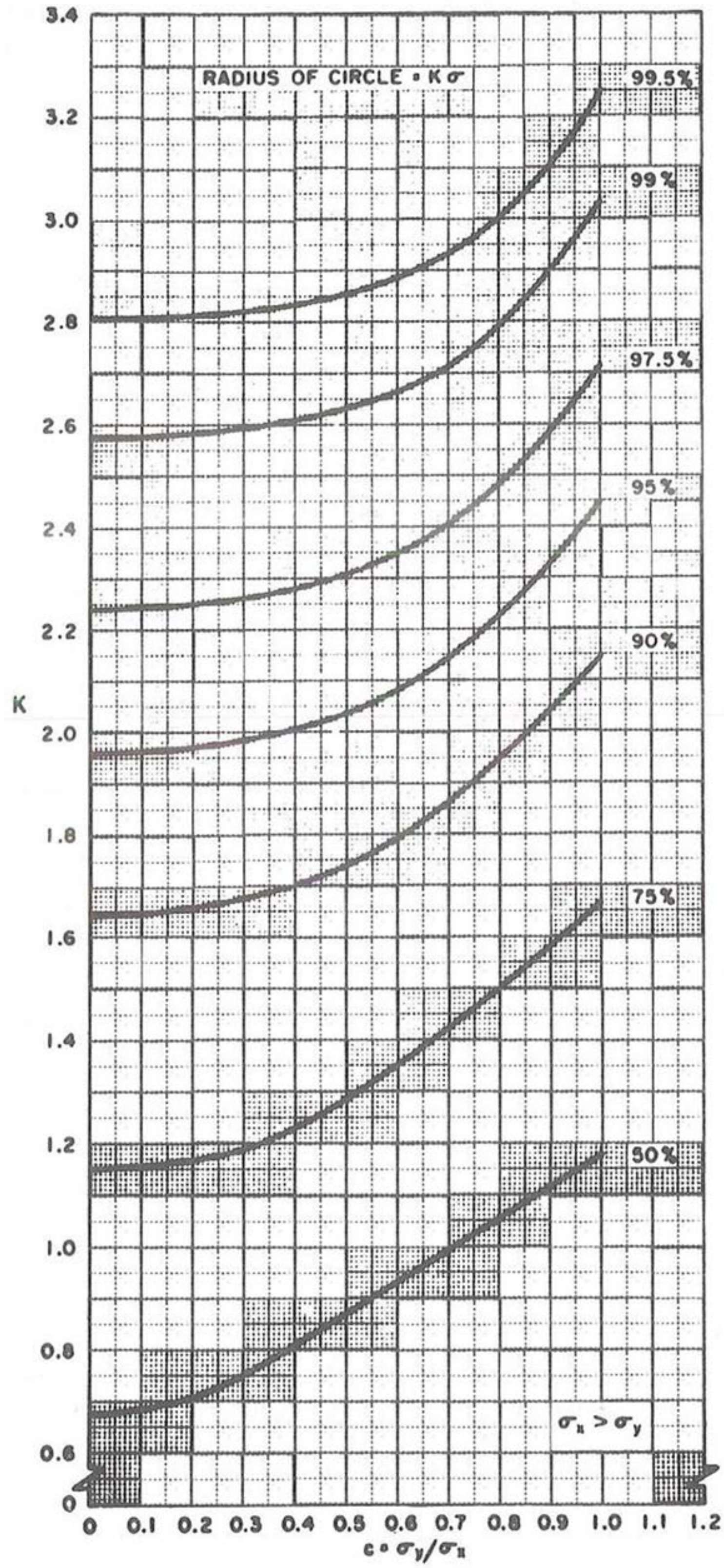


Figure 305g. Factors for conversion of probability ellipse to circle of equivalent probability.

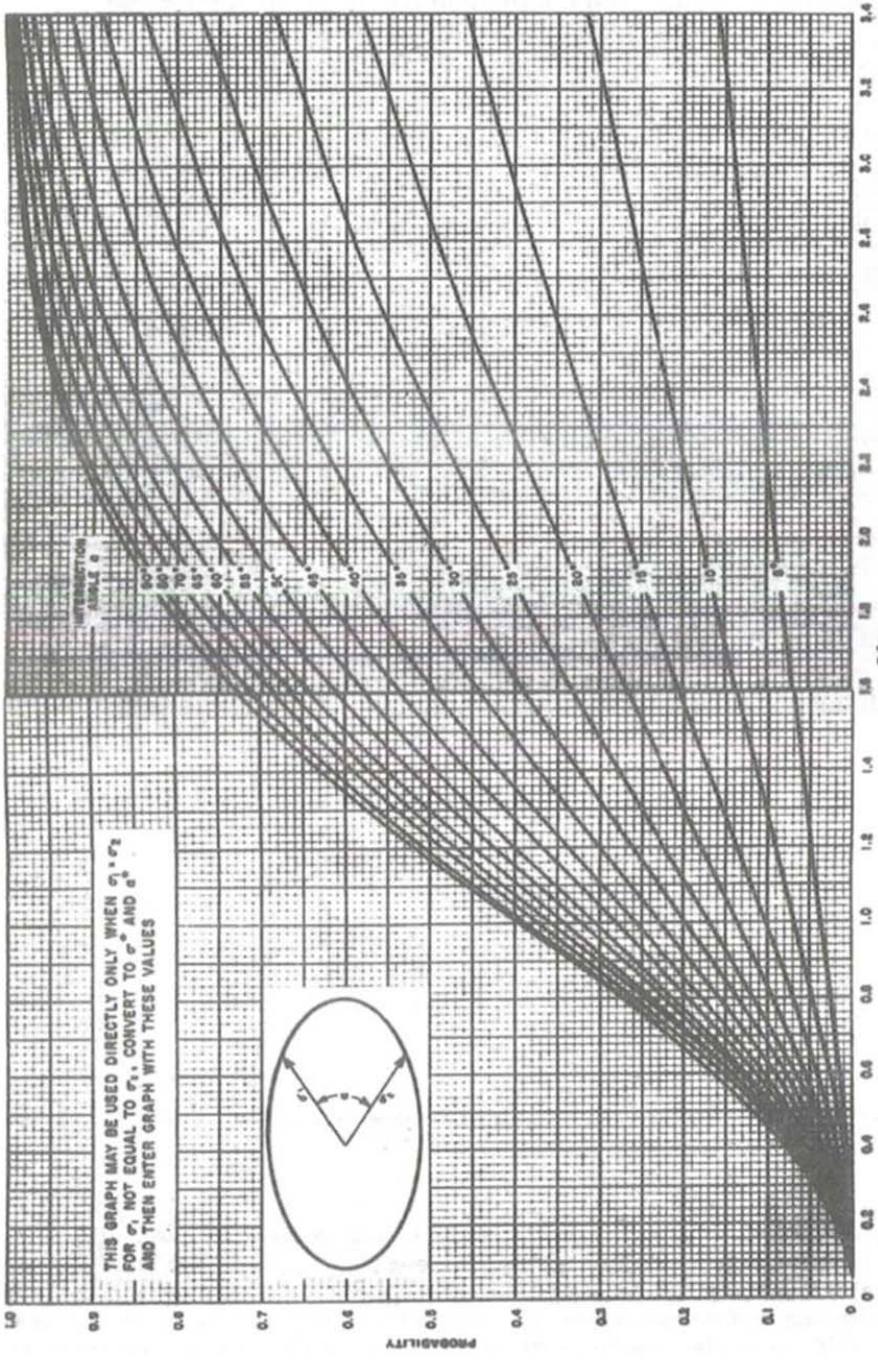


Figure 305h. Probability versus the radius of the circle divided by the standard error and the angle of cut for elliptical bivariate distributions with two equal standards deviations.

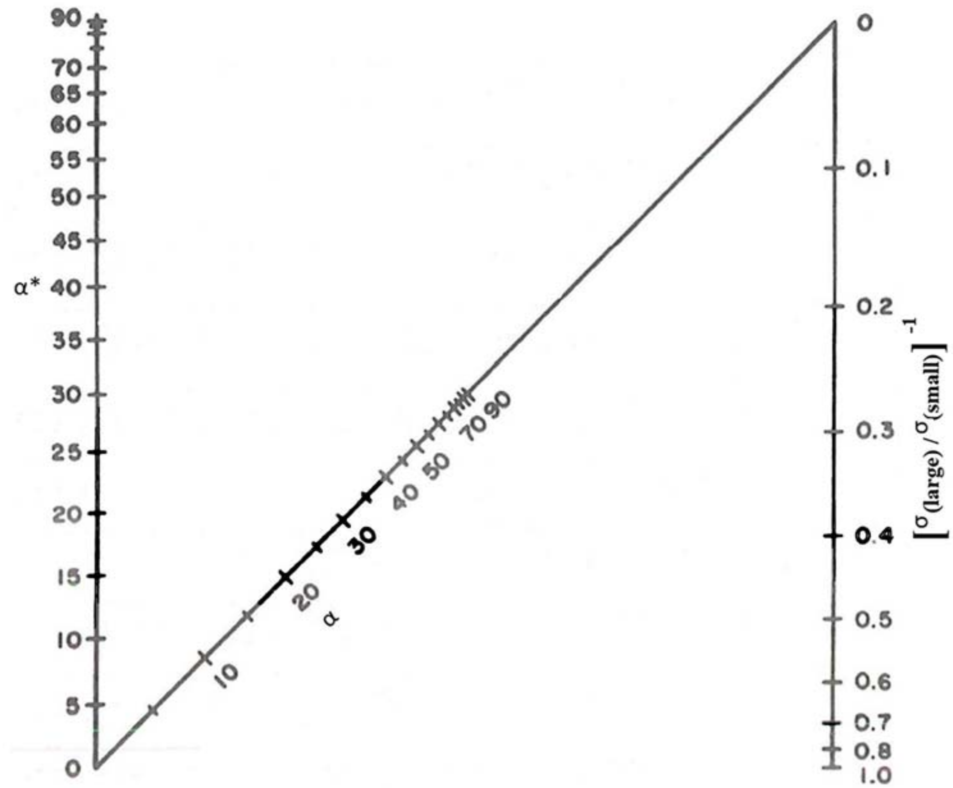


*Figure 305i. Nomogram to obtain α *.*

308. One-Dimensional Errors

The terms **standard deviation**, **sigma** (σ), and **root mean square (RMS) error** have the same meaning in reference to one-dimensional errors. The basic equation of the normal (Gaussian) distribution indicates the use of the Greek letter sigma, σ, from which its use for standard deviation arises:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

where the Greek letter μ is the mean of the distribution.

Standard deviation of a measurement system is a property that may be determined experimentally. If a large number of measurements of the same quantity, a length for example, are made and compared with their mean value, the standard deviation is the square root of the sum of the squares of the differences (deviations) of the measurements from the mean value divided by one less than the number of measurements taken. The mean, or average value, is the sum of the measurements divided by the number of the measurements. Symbolically this operation is represented as:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n-1}}, \quad \mu = \frac{\sum_{i=1}^{n}x_i}{n}$$

The term root-mean-square (RMS) error comes from this latter method of computation.

Numerically, the values between the mean plus or minus one sigma (one standard deviation) corresponds to 68.27 percent of the distribution. That is, if a large number of measurements were made of a given quantity, 68.27 percent of the errors would be within the value of the mean plus or minus one standard deviation, or within $\mu \pm 1\sigma$. Likewise, errors within $\mu \pm 2\sigma$ correspond to 95.45 percent of the total errors and errors within $\mu \pm 3\sigma$ correspond to 99.73 percent of the total errors. Colloquially, these conditions are described as not exceeding the one-, two-, and three-sigma values, respectively.

The term probable error is identical in concept to standard deviation. The term differs from standard deviation in that it refers to the median error; that is, no more than half the errors in the measurement sample are greater than the value of the probable error. Linear probable error is related to standard deviation by a multiplication factor (Table 308a). One probable error equals 0.6745 times one standard deviation.

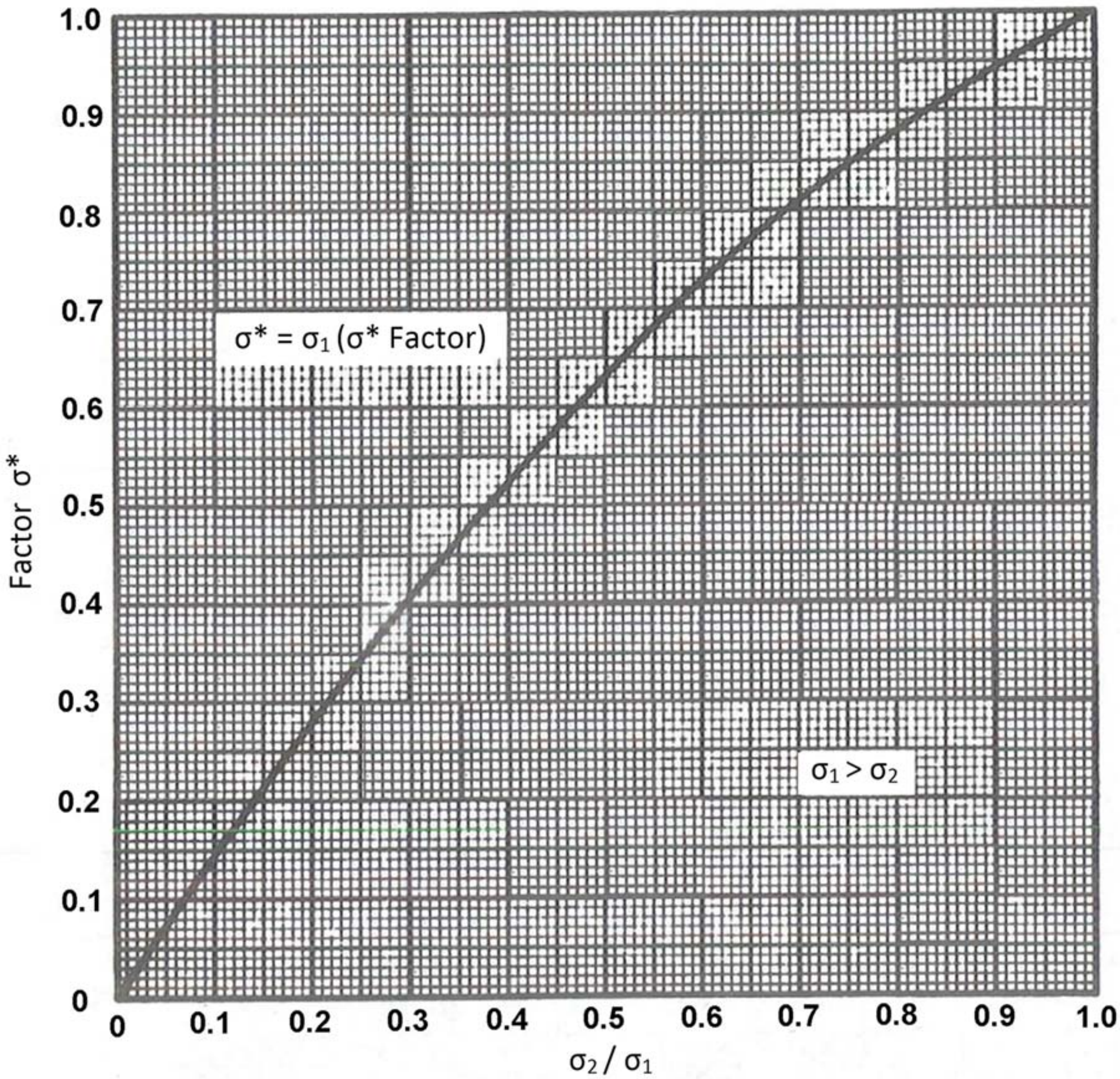


Figure 305j. σ ** factors versus /* σ_2/σ_1 *ratio.*

α	σ_x	σ_y	*c*	*K*	CEP	Error Factor
90	1.0	1.0	1.0	1.177	1.177	1.00
80	1.10	0.924	0.839	1.078	1.186	1.01
70	1.234	0.865	0.700	0.996	1.228	1.042
60	1.414	0.817	0.577	0.914	1.292	1.099
50	1.672	0.782	0.466	0.847	1.420	1.206
45	1.847	0.766	0.414	0.815	1.508	1.281
40	2.06	0.753	0364	0.783	1.620	1.376
30	2.74	0.733	0.268	0.734	2.01	1.710
20	4.06	0.718	0.176	0.700	2.85	2.42
10	8.11	0.710	0.087	0.680	5.52	4.69

Table 306a. Significant parameters of error ellipses when $\sigma_1 = \sigma_2$

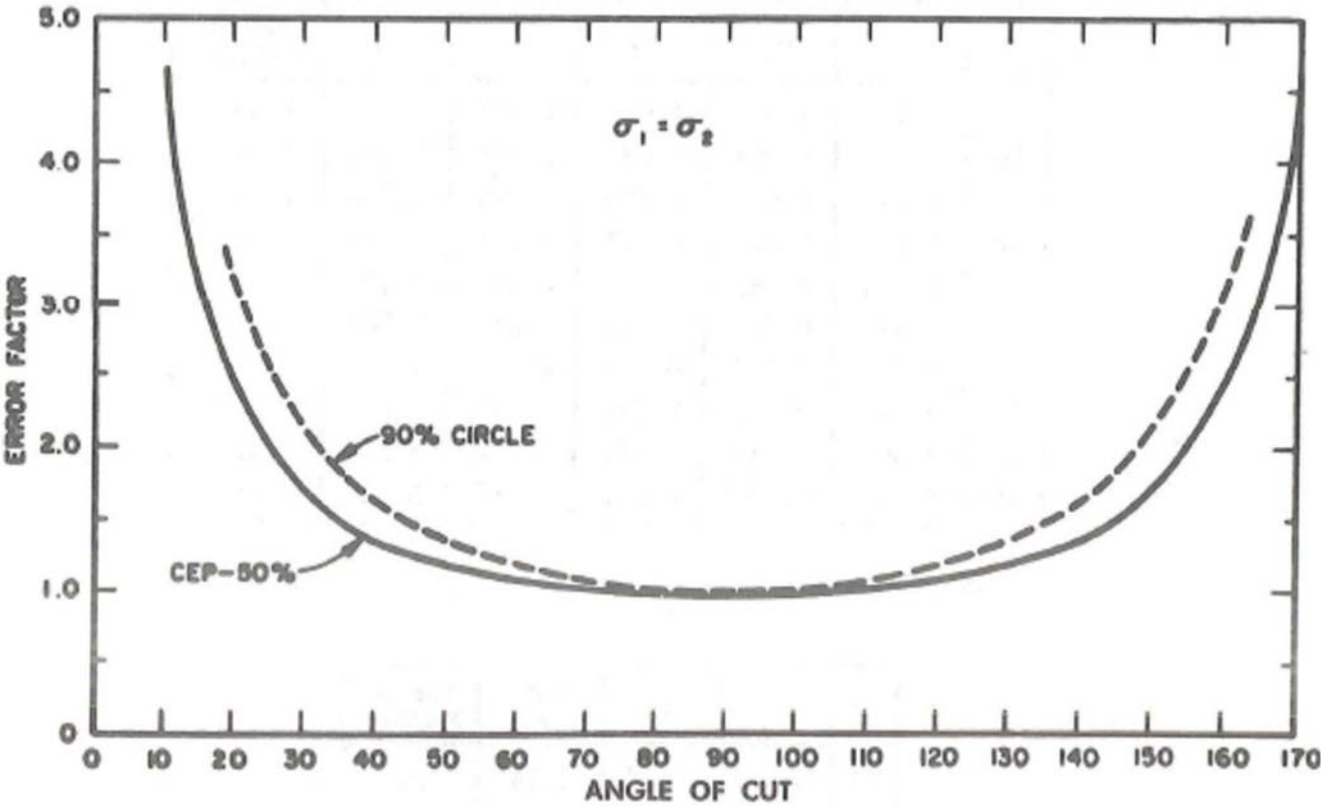


Figure 306b. CEP magnification versus angle of cut.

α	*c*	*K*	90% R	Error Factor
90	1.0	2.145	2.145	1.00
80	0.839	1.98	2.18	1.015
70	0.700	1.86	2.30	1.07
60	0.577	1.775	2.51	1.7
50	0.466	1.72	2.88	1.34
45	0.414	1.702	3.15	1.47
40	0.364	1.687	3.47	1.615
30	0.268	1.665	4.53	2.11
20	0.176	1.652	6.72	3.13
10	0.087	1.645	13.35	6.22

Table 306c. 90 percent error factor

α	*P*	*P*
90	50	90
80	49.4	89.2
70	47.5	86.9
60	44.0	82.4
50	39.5	76
40	37	66
30	25	53
20	17	37
10	8	19

Table 306d. Probability decrease with decreasing angle of cut for a circle of constant radius

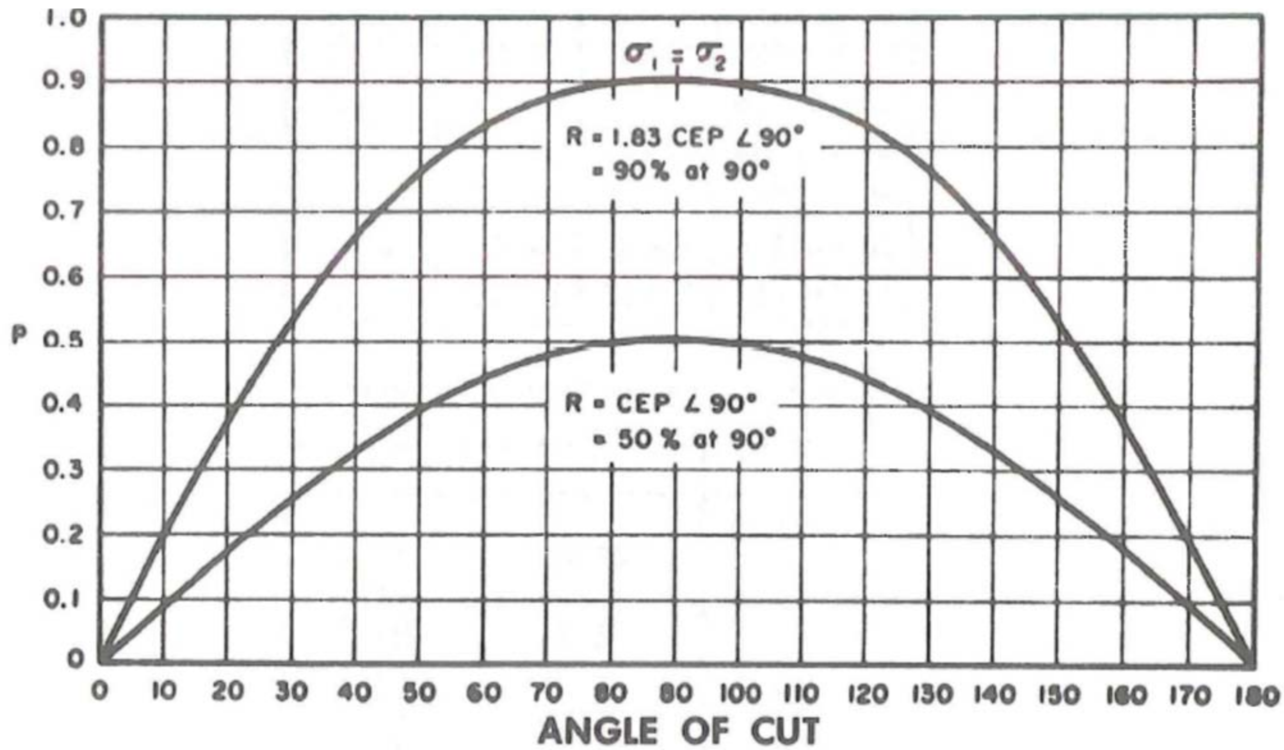


Figure 306e. Decrease in probability for a circle of constant radius versus angle of cut.

The term **variance** is met most frequently in detailed mathematical discussions.

From/To	**50.00%**	**68.27%**	**95.00%**	**99.73%**
50.00%	1.0000	1.4826	2.9059	4.4475
68.27%	0.6745	1.0000	1.9600	3.0000
95.00%	0.3441	0.5102	1.0000	1.5307
99.73%	0.2248	0.3333	0.6533	1.0000

Table 308a. Linear error conversion factors.

309. Two-Dimensional Error

Terms similar or identical in words to those used for one-dimensional error descriptions are also used with two-dimensional or **bivariate error** descriptions. However, in the two-dimensional case, not all of these terms have the same meaning as before; considerable care is needed to avoid confusion.

Standard deviation or **sigma** has a definable meaning only in the specific case of the circular normal distribution where $\sigma_x = \sigma_y$:

$$P_R = 1 - e\frac{R^2}{2\sigma^2}$$

In the case of the circular normal distribution, the standard deviation σ is equivalent to the standard deviation along both orthogonal axes. Because of concern with a radial distribution, the total distribution of errors involves numbers different from those of the linear case (Table 308a and Table 309a). In the circular case, 1σ error indicates that 39.35 percent of the errors would not exceed the value of the 1σ error; 86.47 percent would not exceed the 2σ error; 98.89 percent would not exceed the 3σ error; and 99.78 percent would not exceed the 3.5σ error.

From/To	**39.35%**	**50.00%**	**63.21%**	**95.00%**	**99.78%**
39.35%	1.0000	1.1774	1.4142	2.4477	3.5000
50.00%	0.8493	1.0000	1.2011	2.0789	2.9726
63.21%	0.7071	0.8325	1.0000	1.7308	2.4749
95.00%	0.4085	0.4810	0.5778	1.0000	1.4299

Table 309a. Circular error conversion factors.

From/To	**39.35%**	**50.00%**	**63.21%**	**95.00%**	**99.78%**
99.78%	0.2857	0.3364	0.4040	0.6993	1.0000

Table 309a. Circular error conversion factors.

Because the usual case where there are two-dimensional distributions is that the standard deviations are different, resulting in an elliptical distribution, the circular standard deviation is less useful than the linear standard deviation. It is more common to describe two-dimensional distributions by the two separate one-dimensional standard deviations associated with each error axis. References, however, often do not make this distinction, referring to the position accuracy of a system as 600 feet (2σ), for example. Such a description should leave the reader wondering whether the measure is circular error, in which case the numbers describe the 86 percent probability circle, or whether the number are to be interpreted as one-dimensional sigmas along each axis, in which case the 95 percent probability circle is indicated (assuming the distribution to be circular, which actually it may not be).

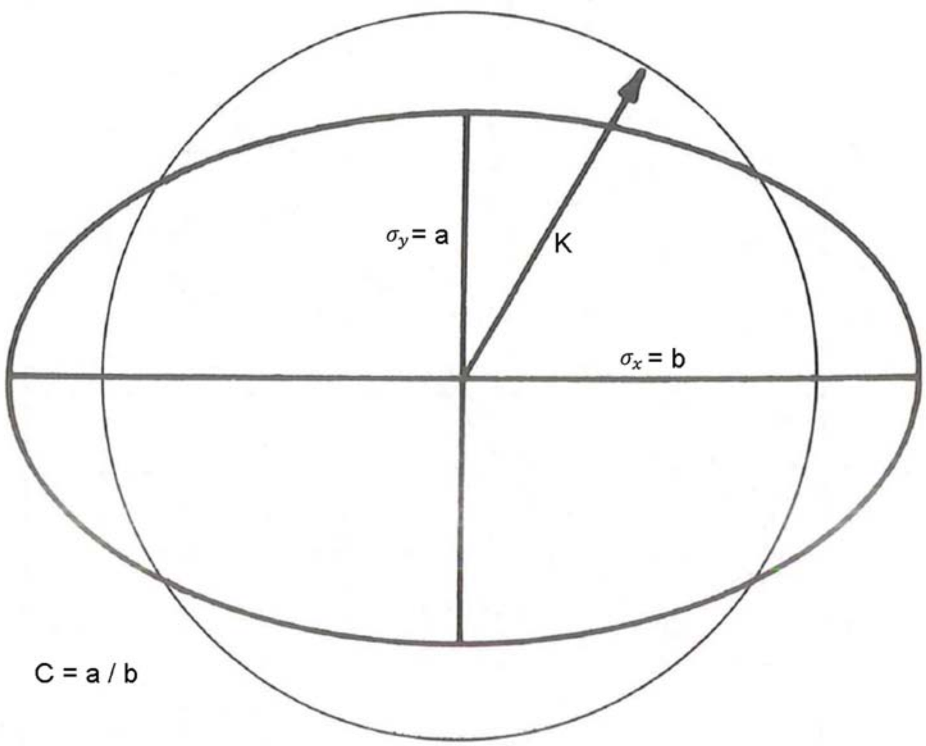


Figure 309b. Error ellipse and circle of equivalent probability.

The term **RMS (root mean square) error** when applied to two-dimensional errors does not have the same meaning as standard deviation. The term has the same meaning as radial error or d_{rms}, discussed later. Such use of the term is deprecated.

In a circular normal distribution, the term **circular probable error (CPE)** or **circular error probable (CEP)** refers to the radius of the circle inside of which there is a 50 percent probability of being located.

The term CEP is also used to indicate the radius of a circle inside of which there is a 50 percent probability of being located, even though the actual error figure (Figure 309b) is an ellipse. Article 305 describes one of the methods of obtaining such CEP equivalents when given ellipses of varying eccentricities. Curves and tables are available for performing this calculation. Despite the availability of these curves and tables, approximations are often made for this calculation of a CEP when the actual error distribution is elliptical. Several of these approximations are indicated and plotted for comparison with the exact curve in Figure 309c. Of the various approximations shown, the top curve, the one which diverges the most rapidly, appears to be the most commonly used.

Another factor of interest concerning the relationship of the CEP to various ellipses is that the area of the CEP circle is always greater than the basic ellipse. Table 309d indicates that the divergence between the actual area of the ellipse of interest and the circle of equivalent probability increases as the ellipse becomes thinner and more elongated.

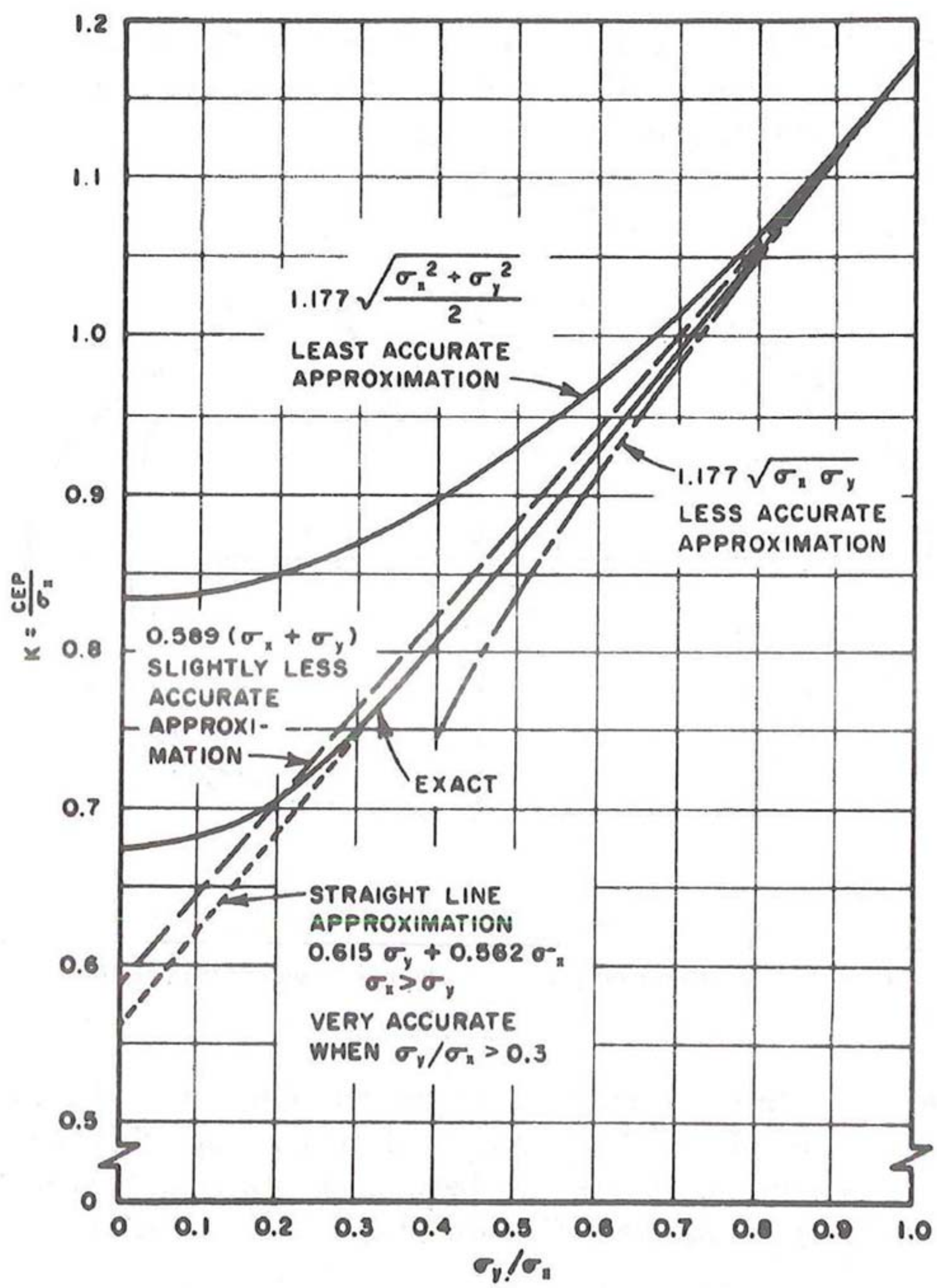


Figure 309c. CEP for elliptical error distribution approximations.

C = a / b	Area of 50% ellipse	Area of equivalent circle
0.0	0	1.43
0.1	0.437	1.46
0.2	0.874	1.56
0.3	1.31	1.76
0.4	1.75	2.06

Table 309d. Comparison of areas of 50% ellipses of varying eccentricities with areas of circles of equivalent probabilities.

C = a / b	Area of 50% ellipse	Area of equivalent circle
0.5	2.08	2.37
0.6	2.62	2.74
0.7	3.06	3.12
0.8	3.49	3.52
0.9	3.93	3.94
1.0	4.37	4.37

Table 309d. Comparison of areas of 50% ellipses of varying eccentricities with areas of circles of equivalent probabilities.

The value of the CEP may be related to the radius of other values of probability circles analytically for the case of the circular normal distribution by solving the basic equation for various values of probability. For this special case of the circular normal distribution, these relationships are shown drawn to scale in Figure 309e with the associated values tabulated in Table 309f.

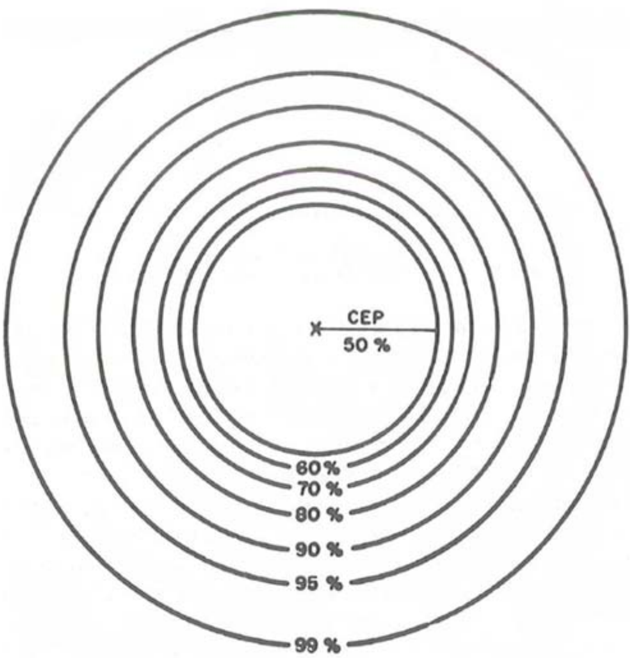


Figure 309e. Relationship between CEP and other probability circles.

Multiply values of CEP by	To obtain radii of circle of probability
1.150	60%
1.318	70%

Table 309f. Relationship between CEP and radii of other probabilities circles of the circular normal distribution.

Multiply values of CEP by	To obtain radii of circle of probability
1.414	75%
1.524	80%
1.655	85%
1.823	90%
2.079	95%
2.578	99%

Table 309f. Relationship between CEP and radii of other probabilities circles of the circular normal distribution.

The derivation of these values is shown in the following analysis. First, the factor relating the CEP to the circular sigma is derived, then, as a second example, the relationship between the 75 percent probability circle and the circular sigma is derived. The ratio of these two values is then the value shown in Table 309f for the 75 percent value.

The circular normal distribution equation is:

$$P_R = 1 - e - \frac{R^2}{2\sigma^2} \ ,$$

and

$$CEP = P(R) = 0.5$$

$$1 - e - \frac{R^2}{2\sigma^2} = 0.5$$

$$e - \frac{R^2}{2\sigma^2} = 0.5 \ .$$

Take the natural logarithm of both sides

$$\ln\left(e - \frac{R^2}{2\sigma^2}\right) = \ln\ 0.5$$

$$\frac{R^2}{2\sigma^2} = \ln 2 \qquad (\ln 0.5 = -\ln 2)$$

$$R = 1.1774\sigma .$$

For the 75 percent probability circle,

$$1 - e - \frac{R^2}{2\sigma^2} = 0.75$$

$$e - \frac{R^2}{2\sigma^2} = 0.25$$

$$\ln\left(e - \frac{R^2}{2\sigma^2}\right) = \ln 0.25$$

$$\frac{R^2}{2\sigma^2} = \ln 4$$

$$R = 1.665\sigma$$

$$\frac{R(75\%)}{R(50\%)} = \frac{1.665\sigma}{1.177\sigma} = 1.414 .$$

The factors tabulated in Table 309f are sometimes used to relate varying probability circles when the basic distribution is not circular, but elliptical. That such a procedure is inaccurate may be seen by the curves of . It can be seen that the errors involved are small when the eccentricities are small. But the errors increase significantly when both high values of probability are desired and when the ellipticity increases in the direction of long, narrow distributions.

The terms **radial error, root mean square error,** and $\boldsymbol{d_{rms}}$ are identical in meaning when applied to two-dimensional errors. Figure 309h illustrates the definition of d_{rms}. It is seen to be the square root of the sum of the square of the 1 sigma error components along the major and minor axes of a probability ellipse. The figure details the definition of 1 d_{rms}. Similarly, other values of d_{rms} can be derived by using the corresponding values of sigma. The measure d_{rms} is not equal to the square root of the sum of the squares of σ_1 and σ_2 that are the basic errors associated with the lines of position of a particular measuring system. The procedures described in section 305 must first be utilized to obtain the values shown as σ_x and σ_y.

The three terms (radial error, root-mean-square error, and d_{rms}) used as a measure of error are somewhat confusing because they do not correspond to a fixed value of probability for a given value of the error measure. The terms can be conveniently related to other error measures only when $\sigma_x = \sigma_y$, and the probability figure is a circle. In the more common elliptical cases, the probability associated with a fixed value of d_{rms} varies as a function of the eccentricity of the ellipse. One d_{rms} is defined as the radius of the circle obtained when $\sigma_x = 1$, in Figure 309h, and σ_y varies from 0 to 1. Likewise, 2 d_{rms} is the radius of the circle obtained when $\sigma_x = 2$, and σ_y varies from 0 to 2. Values of the length of the radius d_{rms} can be calculated as shown in Table 309j. From these values the associated

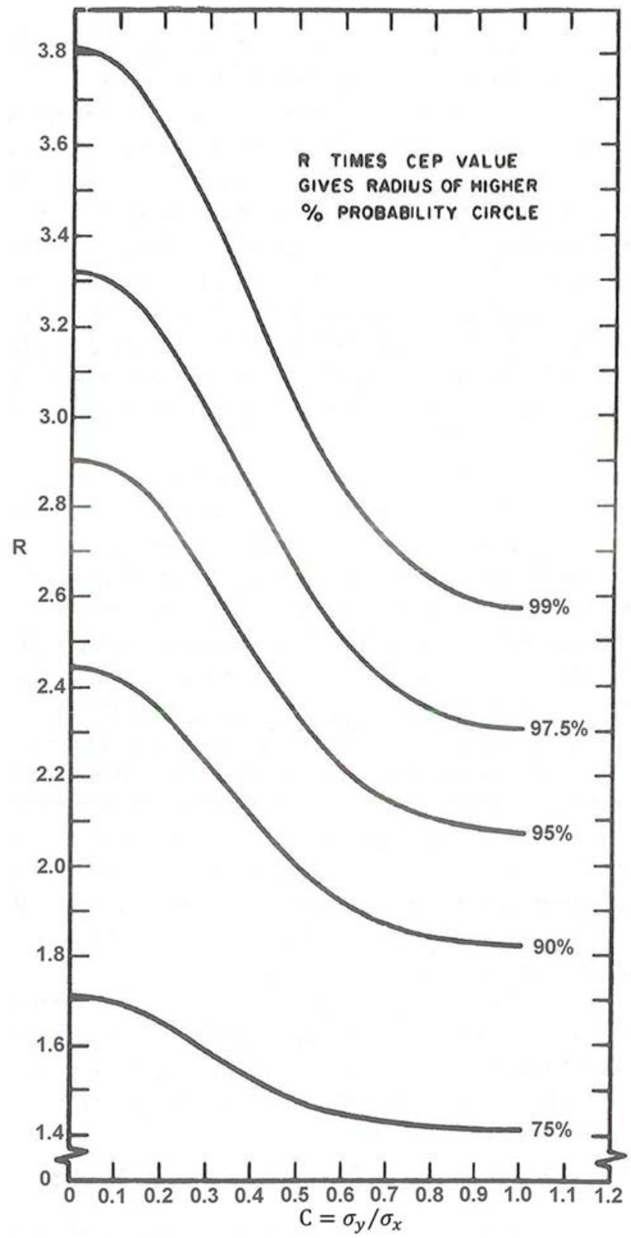


Figure 309g. Relation of probability circles to CEP versus ellipticity.

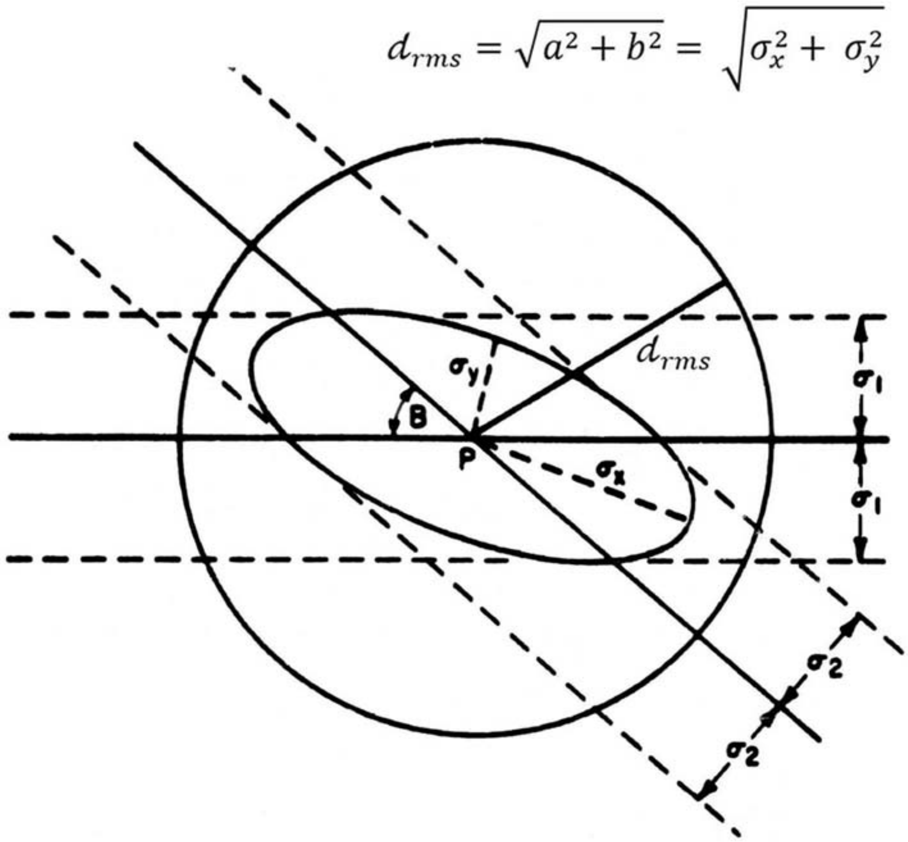


Figure 309h. CEP for elliptical error distribution approximations.

probabilities can be determined from the tables of section 305. The variations of probability associated with the values of 1 d_{rms} and 2 d_{rms} are shown in the curves of and . shows the lack of a constant relationship in a slightly different way. Here the ratio d_{rms}/CEP is plotted against the same measure of ellipticity. The three figures show graphically that there is not a constant value of probability associated with a single value of d_{rms}.

Figure 309i shows the substitution of the circular form for elliptical error distributions. When σ_x and σ_y are equal, the probability represented by 1 d_{rms} is 63.21 percent. When σ_x and σ_y are unequal (σ_x being the greater value), the probability varies from 64 percent when $\sigma_y/\sigma_x = 0.8$ to 68 percent when $\sigma_y/\sigma_x = 0.3$.

310. Navigation System Accuracy

In a navigation system, **predictability** is the measure of the accuracy with which the system can define the position in terms of geographical coordinates; **repeatability** is the measure of the accuracy with which the system permits the user to return to a position as defined only in terms of the coordinates peculiar to that system. **Predictable accuracy**, therefore, is the accuracy of positioning with respect to geographical coordinates; **repeatable accuracy** is the accuracy with which the user can return to a position whose coordinates have been measured previously with the same system. For example, the distance specified for the repeatable accuracy of a system such as GPS is the distance between two GPS positions established using the same satellites at different times. The correlation between the geographical coordinates and the system coordinates may or may not be known.

Relative accuracy is the accuracy with which a user can determine their position relative to that of another user of the same navigation system at the same time. Hence, a system with high relative accuracy provides good rendezvous capability for the users of the system. The correlation between the geographical coordinates and the system coordinates is not relevant.

311. Most Probable Position

Some navigators, particularly those of little experience, have been led by the simplified definitions and explanations usually given in texts to conclude that the line of position is infallible, and that a fix is without error, overlooking the frequent incompatibility of these two notions. Too often the idea has prevailed that information is either all right or all

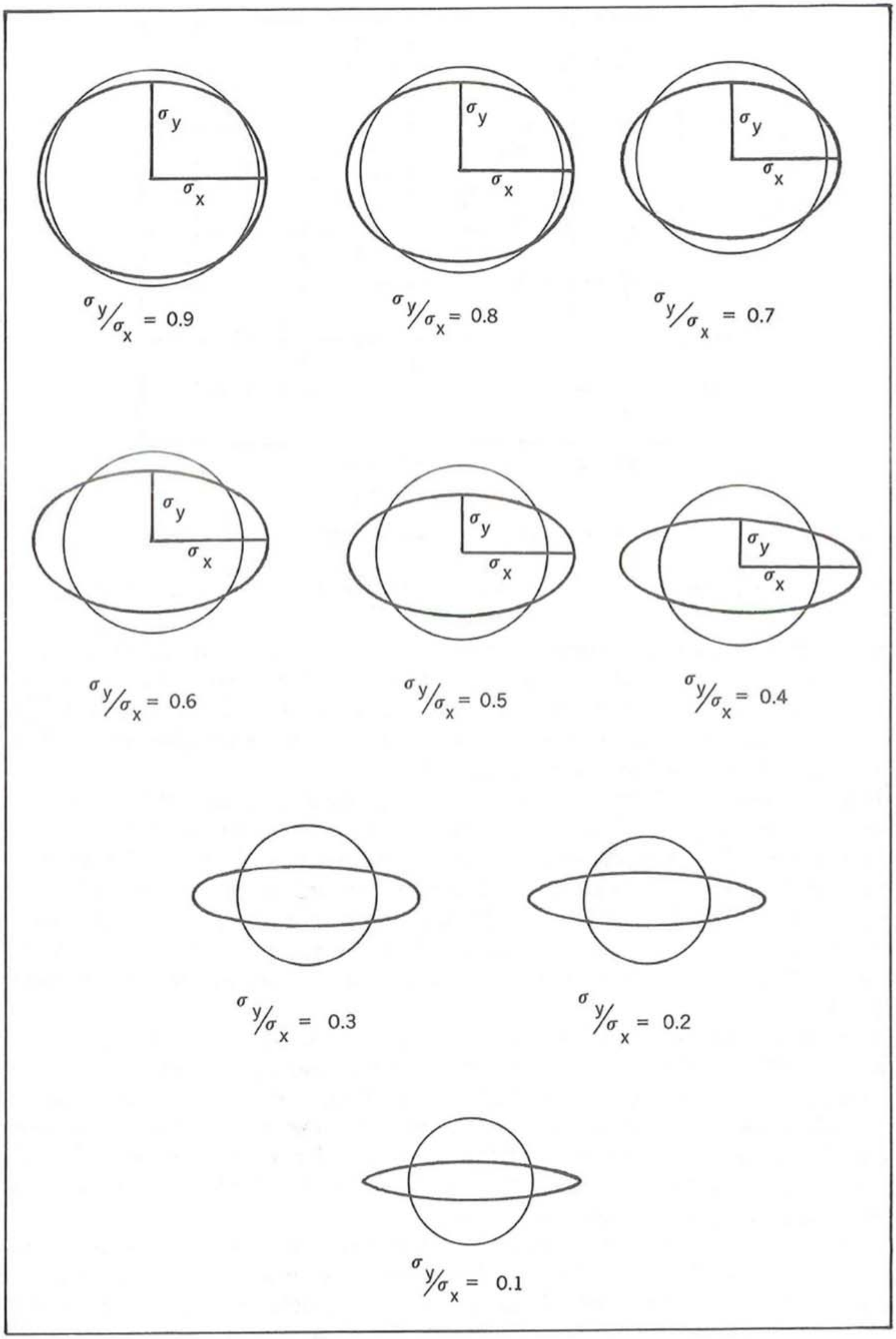


Figure 309i. Substitution of the circular form for elliptical error distributions.

σ_y	σ_x	LENGTH OF 1 d_{rms}	PROBABILITY	
			1 $_{drms}$	2 d_{rms}
0.0	1.0	1.000	0.683	0.954
0.1	1.0	1.005	0.682	0.955
0.2	1.0	1.020	0.682	0.957
0.3	1.0	1.042	0.676	0.961
0.4	1.0	1.077	0.671	0.966
0.5	1.0	1.118	0.662	0.969
0.6	1.0	1.166	0.650	0.973
0.7	1.0	1.220	0.641	0.977
0.8	1.0	1.280	0.635	0.980
0.9	1.0	1.345	0.632	0.981
1.0	1.0	1.414	0.632	0.982
$d_{rms} = \sqrt{\sigma_x^2 + \sigma_y^2}$ when σ_x and σ_y are at right angles to each other.				

Table 309j. Calculations of d_{rms}.

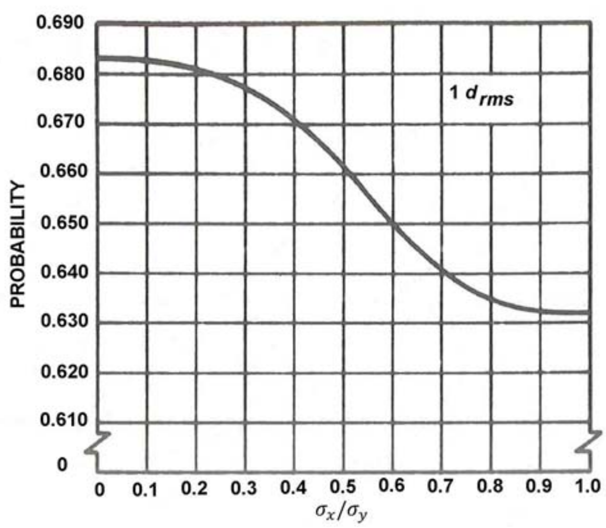


Figure 309k. Variation in d_{rms} with ellipticity (1 d_{rms})..

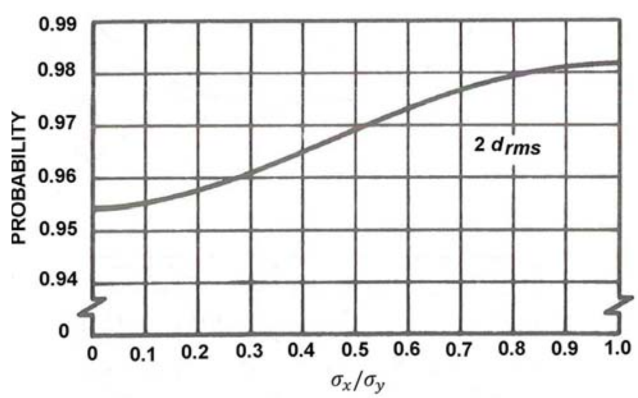


Figure 309l. Variation in d_{rms} with ellipticity (2 d_{rms}).

wrong. An example is the practice of establishing an estimated position at the foot of the perpendicular from a dead reckoning position to a line of position. The assumption is that the vessel *must* be somewhere on the line of position. The limitations of this often valuable practice are not understood by these inexperienced navigators.

A more realistic concept is that of the **most probable position (MPP)**, which recognizes the probability of error in *all* navigational information, and determines position by an evaluation of all available information, using the principles of errors.

Suppose a vessel were to start from a completely accurate position and proceed on dead reckoning. If course and speed over the bottom were of equal accuracy, the uncertainty of dead reckoning positions would increase equally in all directions with either distance or elapsed time (for any one speed these would be directly proportional and therefore either could be used). Therefore, a circle of uncertainty would grow around the dead reckon-

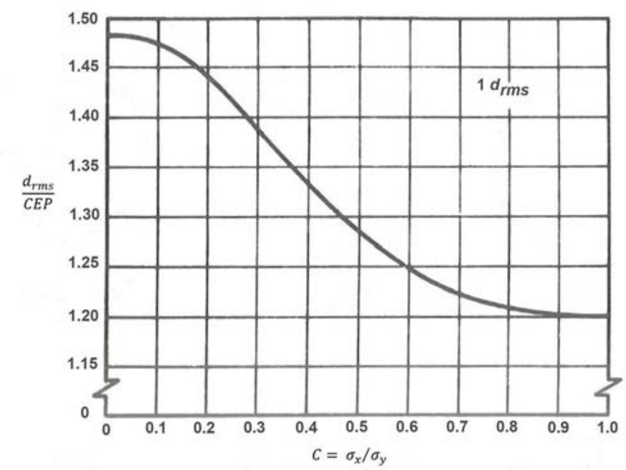


Figure 309m. Ellipticity versus d_{rms} /CEP (1 d_{rms}).

ing position as the vessel proceeded. If the navigator had full knowledge of the distribution and nature of the errors of course and speed, and the necessary knowledge of statistical analysis, s/he could compute the radius of the circle of uncertainty, using the 50 percent, 95 percent, or other probabilities.

In ordinary navigation, this is not practicable, but based upon experience and judgment, the navigator might estimate at any time the likely error of his or her dead reckoning or estimated position. With practice, navigators might acquire considerable skill in making this estimate. They would take into account, too, the fact that the area of uncertainty might be better represented by a circle, the major axis being along the course line if the estimated error of the speed were greater than that of the course, and the minor axis being along the course line if the estimated error of the course were greater. They would recognize, too, that the size of the area of uncertainty would not grow in direct proportion to the distance or elapsed time, because disturbing factors such as wind and current could not be expected to remain of constant magnitude and direction. Also, they would know that the starting point of the dead reckoning would not be completely free from error.

At some future time additional positional information would be obtained. This might be a line of position from a celestial observation. This, too, would be accompanied by an estimated error which might be computed for a certain probability if the necessary information and knowledge were available. If the dead reckoning had started from a good position obtained by means of landmarks, the likely error of the initial position would be very small. At first the dead reckoning or estimated position would probably be more reliable than a line of position obtained by celestial observation. But at *some* distance the two would be equal, and beyond this the line of position might be more accurate.

The determination of most probable position does depend upon *which* information is more accurate. In Figure 311a a dead reckoning position, $\mu_1 = 0.6$, is shown surrounded by a circle of uncertainty with one-sigma error σ_1. A line of position is also shown, with its area of uncertainty with one-sigma error σ_2. The most probable position is within the overlapping area, and if the uncertainty of the dead reckoning position and that of the line of position are about equal, it might be taken at the center of the line perpendicular to the line of position that runs through the dead reckoning position. The intersection of the line of position with the perpendicular is position $\mu_2 = 0.5$. The most probable position means are taken to have only components on the perpendicular. If the overall errors are considered normal, and they are probably approximately, *the effect of each error is proportional to its square, acting on the other position measurement*. Thus, if the likely error of the dead reckoning position is $\sigma_1 = 3$ miles, and that of a line of position is $\sigma_2 =$ 2 miles, the most probable position is nearer the line of position, being given by

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\mu_2 =$$

$$\frac{3^2}{3^2 + 2^2}0.5 + \frac{2^2}{3^2 + 2^2}0.6 = \frac{9}{13}0.5 + \frac{4}{13}0.6 \approx 0.53$$

with an uncertainty given by

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

or

$$\sigma = \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}} = \sqrt{\frac{2^2 + 3^2}{2^2 3^2}} = \sqrt{\frac{13}{36}} \approx 0.60$$

showing that the uncertainty of combining the two position estimates results in a position error smaller than that of either of the two contributing errors.

If a fix is obtained from two lines of position, the area of uncertainty is a circle if the lines are perpendicular, have equal likely errors, and these errors can be considered normal. If one is considered more accurate than the other, the area is an ellipse, the two axes being proportional to the standard deviations of the two lines of position. As shown in Figure 311b, it is also an ellipse if the likely error of each is equal and the lines cross at an oblique angle. If the errors are unequal, the major axis of the ellipse is more nearly in line with the line of position having the smaller likely error.

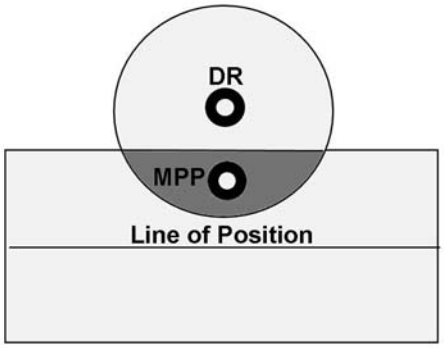


Figure 311a. A most probable position based upon a dead reckoning position and line of position having equal probable errors.

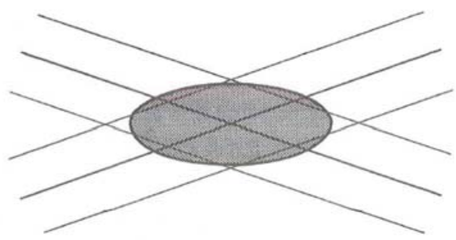


Figure 311b. Ellipse of uncertainty with line of positions of equal probable errors crossing at an oblique angle.

If a fix is obtained from three or more lines of position spread in azimuth by more than 180°, and the error of each line is normal and equal to that of the others, the most probable position is the center of the figure. By "center" is meant that point within the figure which is equidistant from the sides. If the lines are of unequal likely error, the distance of the most probable position from each line of position is proportional to the *square* of the likely error of that line times the sine of the angle formed by the other two lines.

In the discussion of most probable position from lines of position, it has been assumed that no other positional information is available. Usually, this is an incorrect assumption, for there is nearly always a dead reckoning or estimated position. This can be considered in any of several ways. The square of its likely error can be used in the same manner as the square of the likely error of each line of position. A most probable position based upon the dead reckoning or estimated position and the most reliable line of position might be determined as explained above, and that line of position replaced with a new one parallel to it but passing through the most probable position just determined. This adjusted line of position can then be assigned a smaller likely error and used with the other lines of position to determine the overall most probable position. A third way is to establish a likely error for the fix, and consider the most probable position as that point along the straight line joining the fix and the dead reckoning or estimated position, the relative distances being equal to the square of the likely error of each position.

The value of the most probable position determined as suggested above depends upon the degree to which the various errors are in fact normal, and the accuracy with which the likely error of each is established. From a practical standpoint, the second factor is largely a matter of judgment based upon experience. It might seem that interpretation of results and establishment of most probable position is a matter of judgment anyway, and that the procedure outlined above is not needed. If a person will follow this procedure while gaining experience, and evaluate his or her results, the judgment developed should be more reliable than if developed without benefit of knowledge of the principles that are involved. The important point to remember is that the relative effects of normal random errors in any one direction are proportional to their *squares*.

Systematic errors are treated differently. Generally, an attempt is made to discover the errors and eliminate them or compensate for them. In the case of a position determined by three or more lines of position resulting from readings with constant error, the error might be eliminated by finding and applying that correction (including sign) which will bring all lines through a common point.

312. Mistakes

The recognition of a mistake, as contrasted with an error (Section 301), is not always easy, since a mistake may have any magnitude, and may be either positive or negative. A large mistake should be readily apparent if the navigator is alert and has an understanding of the size of error to be reasonably expected. A small mistake is usually not detected unless the work is checked.

If results by two methods are compared, as a dead reckoning position and a line of position, exact agreement is not to be expected. But if the discrepancy is unreasonably large, a mistake is logically suspected. The definition of "unreasonably large" is a matter of opinion. If the 99.9 percent areas of the two results just touch, it is *possible* that no mistake has been made. However, the *probability* of either one having so great an error is remote if the errors are normal. The probability of both having 99.9 percent error of opposite sign at the same instant is very small indeed. Perhaps a reasonable standard is that unless the most accurate result lies within the 95 percent area of the least accurate result, the possibility of a mistake should be investigated. Thus, if the areas of uncertainty shown in Figure 311a represent the 95 percent areas, it is probable that a mistake has been made.

As in other matters pertaining to navigation, judgment is important. The use to be made of the results is certainly a consideration. In the middle of an ocean passage a mistake is usually not serious, and will undoubtedly be corrected before it jeopardizes the safety of the vessel. But if landfall is soon to be made, or if search and rescue operations are to be based upon the position, almost any mistake is intolerable.

313. Conclusion

The correct identification of the nature of an error is important if the error is to be handled intelligently. Thus, the statement is sometimes made that a radio bearing need not be corrected if the receiver is within 50 miles of the transmitter.

The need for a correction arises from the fact that radio waves are assumed to follow great circles, and if radio bearings are to be plotted on a Mercator chart, the equivalent rhumb line is needed. The statement regarding 50 miles implies that the size of the correction is proportional to distance only. It overlooks the fact that latitude and direction of the bearing line are also important factors, and is therefore a dangerous statement unless its limitations are understood.

The recognition of the type of error is also important. A systematic error has quite a different effect than a random error, and cannot be reduced by additional readings unless some method or procedure is instituted which will cause the errors to cancel each other.

The errors for various percentage probabilities are usually of greater interest than the "average" value. The average of a large number of normal errors approaches zero, but the probable (50 percent) error might be quite large.

A person who understands the nature of errors avoids many pitfalls. Thus, the magnitude of the errors of individual lines of position is not a reliable indication of the size of the error of the fix obtained from them. The size of the ·triangle formed by three lines of position has often been used as a guide to the accuracy of the fix, although a large triangle might be the result of a large constant error if the objects observed are equally spaced in azimuth. On the other hand, two lines of position with small errors might produce a fix having a much larger error if the lines cross at a small angle.

314. References

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Greenwalt, C. R. and Shultz, M. E. (1962). *Principles of Error Theory and Cartographic Applications.* Aeronautical Chart and Information Center Technical Report No. 96, St. Louis, Missouri.

CHAPTER 4

CALCULATIONS AND CONVERSIONS

INTRODUCTION

400. Purpose and Scope

This chapter discusses the use of calculators and computers in navigation and summarizes the formulas the navigator depends on during voyage planning, piloting, celestial navigation, and various related tasks. To fully utilize this chapter, the navigator should be competent in basic mathematics including algebra and trigonometry (see Chapter 1 - Mathematics in Volume II) and be familiar with the use of a basic scientific calculator. The navigator should choose a calculator based on personal needs, which may vary greatly from person to person according to individual abilities and responsibilities.

401. Use of Calculators in Navigation

Any common calculator can be used in navigation, even one providing only the four basic arithmetic functions of addition, subtraction, multiplication, and division. Any good scientific calculator can be used for sight reduction, sailings, and other tasks. However, the use computer applications and handheld calculators specifically designed for navigation will greatly reduce the workload of the navigator, reduce the possibility of errors, and assure accuracy of the results calculated.

Calculations of position based on celestial observations have become increasingly uncommon since the advent of GPS as a dependable position reference for all modes of navigation. This is especially true since GPS units provide worldwide positioning with far greater accuracy and reliability than celestial navigation.

However, for those who use celestial techniques, a celestial navigation calculator or computer application can improve celestial position accuracy by easily solving numerous sights, and by reducing mathematical and tabular errors inherent in the manual sight reduction process. They can also provide weighted plots of the LOP's from any number of celestial bodies, based on the navigator's subjective analysis of each sight, and calculate the best fix with latitude/longitude readout.

In using a calculator for any navigational task, it is important to remember that the accuracy of the result, even if carried out many decimal places, is only as good as the least accurate entry. If a sextant observation is taken to an accuracy of only a minute, that is the best accuracy of the final solution, regardless the calculator's ability to solve to 12 decimal places. See Chapter 3 - Navigational Error in Volume II for a discussion of the sources of error in navigation.

Some basic calculators require the conversion of degrees, minutes and seconds (or tenths) to decimal degrees before solution. A good navigational calculator, however, should permit entry of degrees, minutes and tenths of minutes directly, and should do conversions automatically. Though many non-navigational computer programs have an on-screen calculator, they are generally very simple versions with only the four basic arithmetical functions. They are thus too simple for complex navigational problems. Conversely, a good navigational computer program requires no calculator per se, since the desired answer is calculated automatically from the entered data.

The following articles discuss calculations involved in various aspects of navigation.

402. Calculations of Piloting

- **Hull speed in knots** is found by:

  $$S = 1.34\sqrt{\text{waterline length}} \text{ (in feet).}$$

  This is an approximate value which varies with hull shape.

- **Nautical and U.S. survey miles** can be interconverted by the relationships:

  1 nautical mile = 1.15077945 U.S. survey miles.

  1 U.S. survey mile = 0.86897624 nautical miles.

- **The speed of a vessel over a measured mile** can be calculated by the formula:

  $$S = \frac{3600}{T}$$

  where S is the speed in knots and T is the time in seconds.

- **The distance traveled at a given speed** is computed by the formula:

$$D = \frac{ST}{60}$$

where D is the distance in nautical miles, S is the speed in knots, and T is the time in minutes.

- **Distance to the visible horizon in nautical miles** can be calculated using the formula:

$$D = 1.17\sqrt{h_f}\ \text{, or}$$

$$D = 2.07\sqrt{h_m}$$

depending upon whether the height of eye of the observer above sea level is in feet (h_f) or in meters (h_m).

- **Dip of the visible horizon in minutes of arc** can be calculated using the formula:

$$D = 0.97'\sqrt{h_f}\ \text{, or}$$

$$D = 1.76'\sqrt{h_m}$$

depending upon whether the height of eye of the observer above sea level is in feet (h_f) or in meters (h_m)

- **Distance to the radar horizon** in nautical miles can be calculated using the formula:

$$D = 1.22\sqrt{h_f}\ \text{, or}$$

$$D = 2.21\sqrt{h_m}$$

depending upon whether the height of the antenna above sea level is in feet (h_f) or in meters (h_m).

- **Dip of the sea short of the horizon** can be calculated using the formula:

$$Ds = 60\tan^{-1}\left(\frac{h_f}{6076.1\, d_s} + \frac{d_s}{8268}\right)$$

where Ds is the dip short of the horizon in minutes of arc; h_f is the height of eye of the observer above sea level, in feet and d_s is the distance to the waterline of the object in nautical miles.

- **Distance by vertical angle between the waterline and the top of an object** is computed by solving the right triangle formed between the observer, the top of the object, and the waterline of the object by simple trigonometry. This assumes that the observer is at sea level, the Earth is flat between observer and object, there is no refraction, and the object and its waterline form a right angle. For most cases of practical significance, these assumptions produce no large errors.

$$D = \sqrt{\frac{\tan^2 a}{0.0002419^2} + \frac{H-h}{0.7349}} - \frac{\tan a}{0.0002419}$$

where D is the distance in nautical miles, a is the corrected vertical angle, H is the height of the top of the object above sea level, and h is the observer's height of eye in feet. The constants (0.0002419 and 0.7349) account for refraction.

403. Tide Calculations

- **The rise and fall of a diurnal tide** can be roughly calculated from the following table, which shows the fraction of the total range the tide rises or falls during flood or ebb.

Hour	Amount of flood/ebb
1	1/12
2	2/12
3	3/12
4	3/12
5	2/12
6	1/12

404. Calculations of Celestial Navigation

Unlike sight reduction by tables, sight reduction by calculator permits the use of nonintegral values of latitude of the observer, and LHA and declination of the celestial body. Interpolation is not needed, and the sights can be readily reduced from any assumed position. Simultaneous, or nearly simultaneous, observations can be reduced using a single assumed position. Using the observer's DR or MPP for the assumed longitude usually provides a better representation of the circle of equal altitude, particularly at high observed altitudes.

- **The dip correction** is computed in the *Nautical Almanac* using the formula:

$$D = 0.97\sqrt{h}$$

where dip is in minutes of arc and h is height of eye in feet. This correction includes a factor for refraction. The *Air Almanac* uses a different formula intended for air navigation. The differences are of no significance in practical navigation.

- **The computed altitude** (Hc) is calculated using the basic formula for solution of the undivided navigational triangle:

  $$\sin h = \sin L \sin d + \cos L \cos d \cos LHA,$$

  in which h is the altitude to be computed (Hc), L is the latitude of the assumed position, d is the declination of the celestial body, and LHA is the local hour angle of the body. Meridian angle (t) can be substituted for LHA in the basic formula.

  Restated in terms of the inverse trigonometric function:

  $$Hc = \sin^{-1}[(\sin L \ \sin d) + (\cos L \cos d \ \cos LHA)].$$

  When latitude and declination are of contrary name, declination is treated as a negative quantity. No special sign convention is required for the local hour angle, as in the following azimuth angle calculations.

- **The azimuth angle** (Z) can be calculated using the altitude azimuth formula if the altitude is known. The formula stated in terms of the inverse trigonometric function is:

  $$Z = \cos^{-1}\left(\frac{\sin d - (\sin L \sin Hc)}{(\cos L \cos Hc)}\right)$$

  If the altitude is unknown or a solution independent of altitude is required, the azimuth angle can be calculated using the time azimuth formula:

  $$Z = \tan^{-1}\left(\frac{\sin LHA}{(\cos L \tan d) - (\sin L \cos LHA)}\right)$$

  The sign conventions used in the calculations of both azimuth formulas are as follows: (1) if latitude and declination are of contrary name, declination is treated as a negative quantity; (2) if the local hour angle is greater than 180°, it is treated as a negative quantity.

  If the azimuth angle as calculated is negative, add 180° to obtain the desired value.

- **Amplitudes** can be computed using the formula:

  $$A = \sin^{-1}(\sin d \sec L)$$

  this can be stated as

  $$A = \sin^{-1}\left(\frac{\sin d}{\cos L}\right)$$

  where A is the arc of the horizon between the prime vertical and the body, L is the latitude at the point of observation, and d is the declination of the celestial body.

405. Calculations of the Sailings

- **Plane sailing** is based on the assumption that the meridian through the point of departure, the parallel through the destination, and the course line form a plane right triangle, as shown in Figure 405.

  From this: $\cos C = \frac{l}{D}$, $\sin C = \frac{p}{D}$, and $\tan C = \frac{p}{l}$.

  From this: $l = D \cos C$, $D = l \sec C$, and $p = D \sin C$.

  From this, given course and distance (C and D), the difference of latitude (l) and departure (p) can be found, and given the latter, the former can be found, using simple trigonometry. See Chapter 12 - The Sailings, Volume I.

- **Traverse sailing** combines plane sailings with two or more courses, computing course and distance along a series of rhumb lines. See Chapter 12 - The Sailings, Volume I.

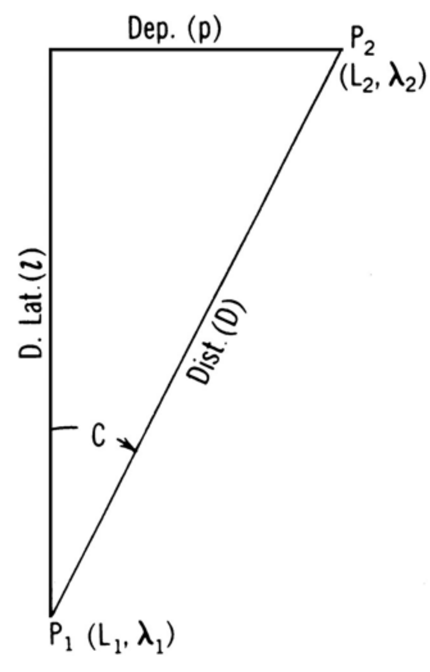

Figure 405. The plane sailing triangle.

- **Parallel sailing** consists of interconverting departure and difference of longitude. Refer to Figure 405.

  DLo = p sec L, and p= DLo cos L

- **Mid-latitude sailing** combines plane and parallel sailing, with certain assumptions. The mean latitude (Lm) is half of the arithmetical sum of the latitudes of two places on the same side of the equator. For places on opposite sides of the equator, the N and S portions are solved separately.

In mid-latitude sailing:

$$DLo = p \sec Lm, \text{ and } p = DLo \cos Lm$$

- **Mercator Sailing** problems are solved graphically on a Mercator chart. For mathematical Mercator solutions the formulas are:

$$\tan C = \frac{DLo}{m} \text{ or } DLo = m \tan C$$

where m is the meridional part from Table 6 in the Tables Part of this volume. Following solution of the course angle by Mercator sailing, the distance is by the plane sailing formula:

$$D = l \sec C.$$

- **Great-circle solutions for distance and initial course angle** can be calculated from the formulas:

$$D = \cos^{-1}[(\sin L_1 \sin L_2 + \cos L_1 \cos L_2 \cos DLo)],$$

and

$$C = \tan^{-1}\left(\frac{\sin DLo}{(\cos L_1 \tan L_2) - (\sin L_1 \cos DLo)}\right)$$

where D is the great-circle distance, C is the initial great-circle course angle, L_1 is the latitude of the point of departure, L_2 is the latitude of the destination, and DLo is the difference of longitude of the points of departure and destination. If the name of the latitude of the destination is contrary to that of the point of departure, it is treated as a negative quantity.

- **The latitude of the vertex**, L_v, is always numerically equal to or greater than L_1 or L_2. If the initial course angle C is less than 90°, the vertex is toward L_2, but if C is greater than 90°, the nearer vertex is in the opposite direction. The vertex nearer L_1 has the same name as L_1.

The latitude of the vertex can be calculated from the formula:

$$L_v = \cos^{-1}(\cos L_1 \sin C)$$

The difference of longitude of the vertex and the point of departure (DLo_v) can be calculated from the formula:

$$DLo_v = \sin^{-1}\left(\frac{\cos C}{\sin L_v}\right).$$

The distance from the point of departure to the vertex (D_v) can be calculated from the formula:

$$D_v = \sin^{-1}(\cos L_1 \sin DLo_v).$$

- **The latitudes of points on the great-circle track** can be determined for equal DLo intervals each side of the vertex (DLo_{vx}) using the formula:

$$L_x = \tan^{-1}(\cos DLo_{vx} \tan L_v)$$

The DLo_v and D_v of the nearer vertex are never greater than 90°. However, when L_1 and L_2 are of contrary name, the other vertex, 180° away, may be the better one to use in the solution for points on the great-circle track if it is nearer the mid point of the track.

The method of selecting the longitude (or DLo_{vx}), and determining the latitude at which the great-circle crosses the selected meridian, provides shorter legs in higher latitudes and longer legs in lower latitudes. Points at desired distances or desired equal intervals of distance on the great-circle from the vertex (D_{vx}) can be calculated using the formulas:

$$L_x = \sin^{-1}[\sin L_v \cos D_{vx}],$$

and

$$DLo_{vx} = \sin^{-1}\left(\frac{\sin D_{vx}}{\cos L_x}\right).$$

A calculator which converts rectangular to polar coordinates provides easy solutions to plane sailings. However, the user must know whether the difference of latitude corresponds to the calculator's X-coordinate or to the Y-coordinate.

406. Calculations of Meteorology and Oceanography

- **Converting thermometer scales** between centigrade, Fahrenheit, and Kelvin scales can be done using the following formulas:

$$C° = \frac{5(F° - 32°)}{9},$$

$$F° = \frac{9}{5}C° + 32°, \text{ and}$$

$$K° = C° + 273.15°.$$

- **Maximum length of sea waves** can be found by the formula:

$$W = 1.5\sqrt{\text{fetch in nautical miles}}\ .$$

- **Wave height** = 0.026 S^2 where S is the wind speed in knots.

- **Wave speed** in knots

$$= 1.34\sqrt{\text{wavelength in feet}},\ \text{or}$$

$$= 3.03\ \text{wave period in seconds}.$$

UNIT CONVERSION

Use the conversion tables that appear on the following pages to convert between different systems of units. Conversions followed by an asterisk * are exact relationships.

MISCELLANEOUS DATA

Area

1 square inch	= 6.4516 square centimeters*
1 square foot	= 144 square inches*
	= 0.09290304 square meter*
	= 0.000022957 acre
1 square yard	= 9 square feet*
	= 0.83612736 square meter
1 square (statute) mile	= 27,878,400 square feet*
	= 640 acres*
	= 2.589988110336 square kilometers*
1 square centimeter	= 0.1550003 square inch
	= 0.00107639 square foot
1 square meter	= 10.76391 square feet
	= 1.19599005 square yards
1 square kilometer	= 247.1053815 acres
	= 0.38610216 square statute mile
	= 0.29155335 square nautical mile

Astronomy

1 mean solar unit	= 1.00273791 sidereal units
1 sidereal unit	= 0.99726957 mean solar units
1 microsecond	= 0.000001 second*
1 second	= 1,000,000 microseconds*
	= 0.01666667 minute
	= 0.00027778 hour
	= 0.00001157 day
1 minute	= 60 seconds*
	= 0.01666667 hour
	= 0.00069444 day
1 hour	= 3,600 seconds*
	= 60 minutes*
	= 0.04166667 day
1 mean solar day	= $24^h03^m56^s.55536$ of mean sidereal time
	= 1 rotation of Earth with respect to Sun (mean)*
	= 1.00273791 rotations of Earth with respect to vernal equinox (mean)
	= 1.0027378118868 rotations of Earth with respect to stars (mean)
1 mean sidereal day	= $23^h56^m04^s09054$ of mean solar time
1 sidereal month	= 27.321661 days
	= $27^d07^h43^m11^s.5$
1 synodical month	= 29.530588 days
	= $29^d12^h44^m02^s.8$

1 tropical (ordinary) year _ _ _ _ _ _ _ _ _ _	= 31,556,925.975 seconds = 525,948.766 minutes = 8,765.8128 hours = $365^{d}.24219879 - 0^{d}.0000000614(t–1900)$, where *t* = the year (date) = $365^{d}05^{h}48^{m}46^{s}$ (–) $0^{s}.0053(t–1900)$
1 sidereal year _ _ _ _ _ _ _ _ _ _ _ _ _ _	= $365^{d}.25636042 + 0.0000000011(t–1900)$, where *t* = the year (date) = $365^{d}06^{h}09^{m}09^{s}.5$ (+) $0^{s}.0001(t–1900)$
1 calendar year (common) _ _ _ _ _ _ _ _ _ _	= 31,536,000 seconds* = 525,600 minutes* = 8,760 hours* = 365 days*
1 calendar year (leap) _ _ _ _ _ _ _ _ _ _ _	= 31,622,400 seconds* = 527,040 minutes* = 8,784 hours* = 366 days
1 light-year _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 9,460,000,000,000 kilometers = 5,880,000,000,000 statute miles = 5,110,000,000,000 nautical miles = 63,240 astronomical units = 0.3066 parsecs
1 parsec _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 30,860,000,000,000 kilometers = 19,170,000,000,000 statute miles = 16,660,000,000,000 nautical miles = 206,300 astronomical units = 3.262 light years
1 astronomical unit _ _ _ _ _ _ _ _ _ _ _ _	= 149,600,000 kilometers = 92,960,000 statute miles = 80,780,000 nautical miles = $499^{s}.012$ light-time = mean distance, Earth to Sun
Mean distance, Earth to Moon _ _ _ _ _ _ _	= 384,400 kilometers = 238,855 statute miles = 207,559 nautical miles
Mean distance, Earth to Sun _ _ _ _ _ _ _ _	= 149,600,000 kilometers = 92,957,000 statute miles = 80,780,000 nautical miles = 1 astronomical unit
Sun's diameter _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 1,392,000 kilometers = 865,000 statute miles = 752,000 nautical miles
Sun's mass _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 1,987,000,000,000,000,000,000,000,000,000,000 grams = 2,200,000,000,000,000,000,000,000,000 short tons = 2,000,000,000,000,000,000,000,000,000 long tons
Speed of Sun relative to neighboring stars _ _	= 19.4 kilometers per second = 12.1 statute miles per second = 10.5 nautical miles per second
Orbital speed of Earth _ _ _ _ _ _ _ _ _ _ _	= 29.8 kilometers per second = 18.5 statute miles per second = 16.1 nautical miles per second
Obliquity of the ecliptic _ _ _ _ _ _ _ _ _ _ _	= $23°27'08''.26 - 0''.4684$ (*t*–1900), where *t* = the year (date)
General precession of the equinoxes _ _ _ _ _	= $50''.2564 + 0''.000222$ (*t*–1900), per year, where *t* = the year (date)
Precession of the equinoxes in right ascension _	= $46''.0850 + 0''.000279$ (*t*–1900), per year, where *t* = the year (date)
Precession of the equinoxes in declination _ _	= $20''.0468 - 0''.000085$ (*t*–1900), per year, where *t* = the year (date)

Magnitude ratio	= 2.512
	$= \sqrt[5]{100}$*

Charts

Nautical miles per inch	= reciprocal of natural scale 72,913.39
Statute miles per inch	= reciprocal of natural scale 63,360*
Inches per nautical mile	= 72,913.39 natural scale
Inches per statute mile	= 63,360 natural scale*
Natural scale	= 1:72,913.39 nautical miles per inch
	= 1:63,360 statute miles per inch*

Earth

Acceleration due to gravity (standard)	= 980.665 centimeters per second per second
	= 32.1740 feet per second per second
Mass-ratio—Sun/Earth	= 332,958
Mass-ratio—Sun/(Earth & Moon)	= 328,912
Mass-ratio—Earth/Moon	= 81.30
Mean density	= 5.517 grams per cubic centimeter
Velocity of escape	= 6.94 statute miles per second
Curvature of surface	= 0.8 foot per nautical mile

World Geodetic System (WGS) Ellipsoid of 1984

Equatorial radius (a)	= 6,378,137 meters
	= 3,443.918 nautical miles
Polar radius (b)	= 6,356,752.314 meters
	= 3432.372 nautical miles
Mean radius (2a + b)/3	= 6,371,008.770 meters
	= 3440.069 nautical miles
Flattening or ellipticity ($f = 1 - b/a$)	= 1/298.257223563
	= 0.003352811
Eccentricity ($e = (2f - f^2)^{1/2}$)	= 0.081819191
Eccentricity squared (e^2)	= 0.006694380

Length

1 inch	= 25.4 millimeters*
	= 2.54 centimeters*
1 foot (U.S.)	= 12 inches*
	= 1 British foot
	= $^1/_3$ yard*
	= 0.3048 meter*
	= $^1/_6$ fathom*
1 foot (U.S. Survey)	= 0.30480061 meter
1 yard	= 36 inches*
	= 3 feet*
	= 0.9144 meter*
1 fathom	= 6 feet*
	= 2 yards*
	= 1.8288 meters*
1 cable	= 720 feet*
	= 240 yards*
	= 219.4560 meters*
1 cable (British)	= 0.1 nautical mile
1 statute mile	= 5,280 feet*
	= 1,760 yards*
	= 1,609.344 meters*
	= 1.609344 kilometers*
	= 0.86897624 nautical mile
1 nautical mile	= 6,076.11548556 feet
	= 2,025.37182852 yards
	= 1,852 meters*
	= 1.852 kilometers*

	= 1.150779448 statute miles
1 meter	= 100 centimeters*
	= 39.370079 inches
	= 3.28083990 feet
	= 1.09361330 yards
	= 0.54680665 fathom
	= 0.00062137 statute mile
	= 0.00053996 nautical mile
1 kilometer	= 3,280.83990 feet
	= 1,093.61330 yards
	= 1,000 meters*
	= 0.62137119 statute mile
	= 0.53995680 nautical mile

Mass

1 ounce	= 437.5 grains*
	= 28.349523125 grams*
	= 0.0625 pound*
	= 0.028349523125 kilogram*
1 pound	= 7,000 grains*
	= 16 ounces*
	= 0.45359237 kilogram*
1 short ton	= 2,000 pounds*
	= 907.18474 kilograms*
	= 0.90718474 metric ton*
	= 0.8928571 long ton
1 long ton	= 2,240 pounds*
	= 1,016.0469088 kilograms*
	= 1.12 short tons*
	= 1.0160469088 metric tons*
1 kilogram	= 2.204623 pounds
	= 0.00110231 short ton
	= 0.0009842065 long ton
1 metric ton	= 2,204.623 pounds
	= 1,000 kilograms*
	= 1.102311 short tons
	= 0.9842065 long ton

Mathematics

π	= 3.14159265358979323846264338327950288 41971
π^2	= 9.8696044011
$\sqrt{\pi}$	= 1.7724538509
Base of Naperian logarithms (e)	= 2.718281828459
Modulus of common logarithms ($\log_{10}e$)	= 0.4342944819032518
1 radian	= 206,264.″80625
	= 3,437′.7467707849
	= 57°.2957795131
	= 57°17′44″.80625
1 circle	= 1,296,000″*
	= 21,600′*
	= 360°*
	= 2π radians*
180°	= π radians*
1°	= 3600″*
	= 60′*
	= 0.0174532925199432957666 radian
1′	= 60″*
	= 0.000290888208665721596 radian
1″	= 0.0000048481368110953599 33 radian
Sine of 1′	= 0.000290888204563424 60
Sine of 1″	= 0.0000048481368110763 7

Meteorology

Atmosphere (dry air)

Nitrogen	= 78.08%	99.99%
Oxygen	= 20.95%	
Argon	= 0.93%	
Carbon dioxide	= 0.03%	
Neon	= 0.0018%	
Helium	= 0.000524%	
Krypton	= 0.0001%	
Hydrogen	= 0.00005%	
Xenon	= 0.0000087%	
Ozone	= 0 to 0.000007% (increasing with altitude)	
Radon	= 0.000000000000000006% (decreasing with altitude)	

Standard atmospheric pressure at sea level	= 1,013.250 dynes per square centimeter
	= 1,033.227 grams per square centimeter
	= 1,033.227 centimeters of water
	= 1,013.250 hectopascals (millibars)*
	= 760 millimeters of mercury
	= 76 centimeters of mercury
	= 33.8985 feet of water
	= 29.92126 inches of mercury
	= 14.6960 pounds per square inch
	= 1.033227 kilograms per square centimeter
	= 1.013250 bars*
Absolute zero	= (–)273.16°C
	= (–)459.69°F

Pressure

1 dyne per square centimeter	= 0.001 hectopascal (millibar)*
	= 0.000001 bar*
1 gram per square centimeter	= 1 centimeter of water
	= 0.980665 hectopascal (millibar)*
	= 0.07355592 centimeter of mercury
	= 0.0289590 inch of mercury
	= 0.0142233 pound per square inch
	= 0.001 kilogram per square centimeter*
	= 0.000967841 atmosphere
1 hectopascal (millibar)	= 1,000 dynes per square centimeter*
	= 1.01971621 grams per square centimeter
	= 0.7500617 millimeter of mercury
	= 0.03345526 foot of water
	= 0.02952998 inch of mercury
	= 0.01450377 pound per square inch
	= 0.001 bar*
	= 0.00098692 atmosphere
1 millimeter of mercury	= 1.35951 grams per square centimeter
	= 1.3332237 hectopascals (millibars)
	= 0.1 centimeter of mercury*
	= 0.04460334 foot of water
	= 0.039370079 inch of mercury
	= 0.01933677 pound per square inch
	= 0.001315790 atmosphere
1 centimeter of mercury	= 10 millimeters of mercury*
1 inch of mercury	= 34.53155 grams per square centimeter
	= 33.86389 hectopascals (millibars)
	= 25.4 millimeters of mercury*
	= 1.132925 feet of water
	= 0.4911541 pound per square inch
	= 0.03342106 atmosphere
1 centimeter of water	= 1 gram per square centimeter
	= 0.001 kilogram per square centimeter
1 foot of water	= 30.48000 grams per square centimeter
	= 29.89067 hectopascals (millibars)
	= 2.241985 centimeters of mercury
	= 0.882671 inch of mercury
	= 0.4335275 pound per square inch
	= 0.02949980 atmosphere

1 pound per square inch	= 68,947.57 dynes per square centimeter
	= 70.30696 grams per square centimeter
	= 70.30696 centimeters of water
	= 68.94757 hectopascals (millibars)
	= 51.71493 millimeters of mercury
	= 5.171493 centimeters of mercury
	= 2.306659 feet of water
	= 2.036021 inches of mercury
	= 0.07030696 kilogram per square centimeter
	= 0.06894757 bar
	= 0.06804596 atmosphere
1 kilogram per square centimeter	= 1,000 grams per square centimeter*
	= 1,000 centimeters of water
1 bar	= 1,000,000 dynes per square centimeter*
	= 1,000 hectopascals (millibars)*

Speed

1 foot per minute	= 0.01666667 foot per second
	= 0.00508 meter per second*
1 yard per minute	= 3 feet per minute*
	= 0.05 foot per second*
	= 0.03409091 statute mile per hour
	= 0.02962419 knot
	= 0.01524 meter per second*
1 foot per second	= 60 feet per minute*
	= 20 yards per minute*
	= 1.09728 kilometers per hour*
	= 0.68181818 statute mile per hour
	= 0.59248380 knot
	= 0.3048 meter per second*
1 statute mile per hour	= 88 feet per minute*
	= 29.33333333 yards per minute
	= 1.609344 kilometers per hour*
	= 1.46666667 feet per second
	= 0.86897624 knot
	= 0.44704 meter per second*
1 knot	= 101.26859143 feet per minute
	= 33.75619714 yards per minute
	= 1.852 kilometers per hour*
	= 1.68780986 feet per second
	= 1.15077945 statute miles per hour
	= 0.51444444 meter per second
1 kilometer per hour	= 0.62137119 statute mile per hour
	= 0.53995680 knot
1 meter per second	= 196.85039340 feet per minute
	= 65.6167978 yards per minute
	= 3.6 kilometers per hour*
	= 3.28083990 feet per second
	= 2.23693632 statute miles per hour
	= 1.94384449 knots
Light in vacuum	= 299,792.5 kilometers per second
	= 186,282 statute miles per second
	= 161,875 nautical miles per second
	= 983.570 feet per microsecond
Light in air	= 299,708 kilometers per second
	= 186,230 statute miles per second
	= 161,829 nautical miles per second
	= 983.294 feet per microsecond
Sound in dry air at 59°F or 15°C and standard sea level pressure	= 1,116.45 feet per second
	= 761.22 statute miles per hour
	= 661.48 knots
	= 340.29 meters per second

Sound in 3.485 percent saltwater at 60°F _ _ _	= 4,945.37 feet per second
	= 3,371.85 statute miles per hour
	= 2,930.05 knots
	= 1,507.35 meters per second

Volume

1 cubic inch_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 16.387064 cubic centimeters*
	= 0.016387064 liter*
	= 0.004329004 gallon
1 cubic foot _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 1,728 cubic inches*
	= 28.316846592 liters*
	= 7.480519 U.S. gallons
	= 6.228822 imperial (British) gallons
	= 0.028316846592 cubic meter*
1 cubic yard_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 46,656 cubic inches*
	= 764.554857984 liters*
	= 201.974026 U.S. gallons
	= 168.1782 imperial (British) gallons
	= 27 cubic feet*
	= 0.764554857984 cubic meter*
1 milliliter _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 0.06102374 cubic inch
	= 0.0002641721 U.S. gallon
	= 0.00021997 imperial (British) gallon
1 cubic meter _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 264.172035 U.S. gallons
	= 219.96878 imperial (British) gallons
	= 35.31467 cubic feet
	= 1.307951 cubic yards
1 quart (U.S.) _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 57.75 cubic inches*
	= 32 fluid ounces*
	= 2 pints*
	= 0.9463529 liter
	= 0.25 gallon*
1 gallon (U.S.)_ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 3,785.412 milliliters
	= 231 cubic inches*
	= 0.1336806 cubic foot
	= 4 quarts*
	= 3.785412 liters
	= 0.8326725 imperial (British) gallon
1 liter _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 1,000 milliliters
	= 61.02374 cubic inches
	= 1.056688 quarts
	= 0.2641721 gallon
1 register ton _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 100 cubic feet*
	= 2.8316846592 cubic meters*
1 measurement ton _ _ _ _ _ _ _ _ _ _ _ _ _	= 40 cubic feet*
	= 1 freight ton*
1 freight ton_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 40 cubic feet*
	= 1 measurement ton*

Volume-Mass

1 cubic foot of seawater _ _ _ _ _ _ _ _ _ _ _	= 64 pounds
1 cubic foot of freshwater _ _ _ _ _ _ _ _ _ _	= 62.428 pounds at temperature of maximum density (4°C = 39°.2F)
1 cubic foot of ice _ _ _ _ _ _ _ _ _ _ _ _ _ _	= 56 pounds
1 displacement ton _ _ _ _ _ _ _ _ _ _ _ _ _	= 35 cubic feet of seawater*
	= 1 long ton

Prefixes to Form Decimal Multiples and Sub-Multiples of International System of Units (SI)

Multiplying factor		Prefix	Symbol
1 000 000 000 000	$= 10^{12}$	tera	T
1 000 000 000	$= 10^{9}$	giga	G
1 000 000	$= 10^{6}$	mega	M
1 000	$= 10^{3}$	kilo	k
100	$= 10^{2}$	hecto	h
10	$= 10^{1}$	deka	da
0. 1	$= 10^{-1}$	deci	d
0. 01	$= 10^{-2}$	centi	c
0. 001	$= 10^{-3}$	milli	m
0. 000 001	$= 10^{-6}$	micro	μ
0. 000 000 001	$= 10^{-9}$	nano	n
0. 000 000 000 001	$= 10^{-12}$	pico	p
0. 000 000 000 000 001	$= 10^{-15}$	femto	f
0. 000 000 000 000 000 001	$= 10^{-18}$	atto	a

NGA MARITIME SAFETY INFORMATION NAUTICAL CALCULATORS

NGA's Maritime Safety Office website offers a variety of online Nautical Calculators for public use. These calculators solve many of the equations and conversions typically associated with marine navigation. See Figure 406.

Figure 406. Link to NGA Nautical Calculators.
https://msi.nga.mil/NGAPortal/MSI.portal?_nfpb=true&_st=&_pageLabel=msi_portal_page_145

List of NGA Maritime Safety information Nautical Calculators https://msi.nga.mil

Celestial Navigation Calculators
Compass Error from Amplitudes Observed on the Visible Horizon
Altitude Correction for Air Temperature
Table of Offsets
Latitude and Longitude Factors
Altitude Corrections for Atmospheric Pressure

List of NGA Maritime Safety information Nautical Calculators https://msi.nga.mil

Altitude Factors & Change of Altitude
Pub 229
Compass Error from Amplitudes observed on the Celestial Horizon
Conversion Calculators
Chart Scales and Conversions for Nautical and Statute Miles
Conversions for Meters, Feet and Fathoms
Distance Calculators
Length of a Degree of Latitude and Longitude
Speed for Measured Mile and Speed, Time and Distance
Distance of an Object by Two Bearings
Distance of the Horizon
Distance by Vertical Angle Measured Between Sea Horizon and Top of Object Beyond Sea Horizon
Traverse Table
Geographic Range
Distance by Vertical Angle Measured Between Waterline at Object and Top of Object
Dip of Sea Short of the Horizon
Distance by Vertical Angle Measured Between Waterline at Object and Sea Horizon Beyond Object
Meridional Parts
Log and Trig Calculators
Logarithmic and Trigonometric Functions
Sailings Calculators
Great Circle Sailing
Mercator NGA Sailing
Time Zones Calculators
Time Zones, Zone Descriptions and Suffixes
Weather Data Calculators
Direction and Speed of True Wind
Correction of Barometer Reading for Height Above Sea Level
Correction of Barometer Reading for Gravity
Temperature Conversions
Relative Humidity and Dew Point
Corrections of Barometer Reading for Temperature
Barometer Measurement Conversions

CHAPTER 5

COMPASS CONVERSIONS

INTRODUCTION

500. Magnetic Compass Error

Directions relative to the northerly direction along a geographic meridian are **true**. In this case, true north is the **reference direction**. If a compass card is horizontal and oriented so that a straight line from its center to 000° points to true north, any direction measured by the card is a true direction and has no error (assuming there is no calibration or observational error). If the card remains horizontal but is rotated so that it points in any other direction, the amount of the rotation is the **compass error**. Stated differently, compass error is the angular difference between true north and **compass north** (the direction north as indicated by a magnetic compass). It is named east or west to indicate the side of true north on which compass north lies.

If a magnetic compass is influenced by no other magnetic field than that of the earth, and there is no instrumental error, its magnets are aligned with the magnetic meridian at the compass, and 000° of the compass card coincides with **magnetic north**. All directions indicated by the card are **magnetic**. As stated in volume I, the angle between geographic and magnetic meridians is called **variation** (**V** or **Var.**). Therefore, if a compass is aligned with the magnetic meridian, compass error and variation are the same.

When a compass is mounted in a vessel, it is generally subjected to various magnetic influences other than that of the earth. These arise largely from induced magnetism in metal decks, bulkheads, masts, stacks, boat davits, guns, etc., and from electromagnetic fields associated with direct current in electrical circuits. Some metal in the vicinity of the compass may have acquired permanent magnetism. The actual magnetic field at the compass is the vector sum, or resultant of all individual fields at that point. Since the direction of this resultant field is generally not the same as that of the earth's field alone, the compass magnets do not lie in the magnetic meridian, but in a direction that makes an angle with it. This angle is called **deviation** (**D** or **Dev.**). Thus, deviation is the angular difference between magnetic north and compass north. It is expressed in angular units and named east or west to indicate the side of magnetic north on which compass north lies. Thus, deviation is the error of the compass in pointing to magnetic north, and all directions measured with compass north as the reference direction are **compass directions**. Since variation and deviation may each be either east or west, the effect of deviation may be to either increase or decrease the error due to variation alone. The algebraic sum of variation and deviation is the total compass error.

For computational purposes, deviation and compass error, like variation, may be designated positive· (+) if east and negative (-) if west.

Variation changes with location. Deviation depends upon the magnetic latitude and also upon the individual vessel, its trim and loading, whether it is pitching or rolling, the heading (orientation of the vessel with respect to the earth's magnetic field), and the location of the compass within the vessel. Therefore, deviation is not published on charts. The effects of variation and deviation on the compass card is depicted in Figure 500.

501. Deviation Table

In practice aboard ship, the deviation is reduced to a minimum through adjustment of the compass. The remaining value, called **residual deviation**, is determined on various headings and recorded in some form of **deviation table**. Figure 502 shows the form used by the United States Navy. This table is entered with the magnetic heading, and the deviation on that heading is determined from the tabulation, separate columns being given for degaussing (DG) equipment off and on. If the deviation is not more than about 2° on any heading, satisfactory results may be obtained by entering the values at intervals of 45° only.

If the deviation is small, no appreciable error is introduced by entering the table with either magnetic or compass heading. If the deviation on some headings is large, the desirable action is to reduce it, but if this is not practicable, a separate deviation table for compass heading entry may be useful. This may be made by applying the tabulated deviation to each entry value of magnetic heading, to find the corresponding compass heading, and then interpolating between these to find the value of deviation at each 15° compass heading.

502. Applying Variation and Deviation

As indicated in Section 500, a single direction may have any of several numerical values depending upon the reference direction used. One should keep clearly in mind the relationship between the various expressions of a direction. Thus, true and magnetic directions differ by the

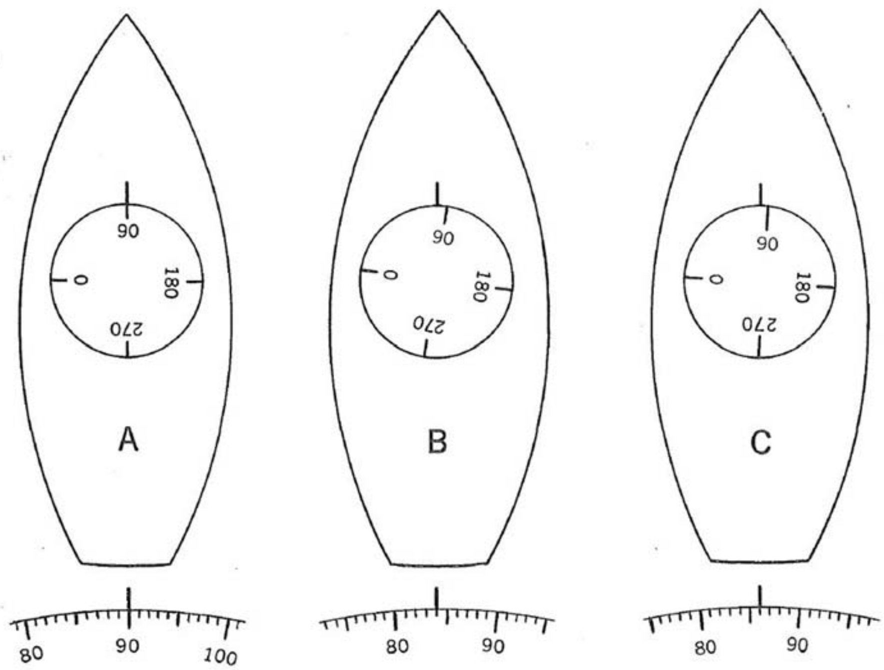

Figure 500. Effects of variation and deviation on the compass card.

variation, magnetic and compass directions differ by the deviation, and true and compass directions differ by the compass error.

If variation or deviation is easterly, the compass card is rotated in a clockwise direction. This brings smaller numbers opposite the lubber's line. Conversely, if either error is westerly, the rotation is counterclockwise and larger numbers are brought opposite the lubber's line. Thus, if the heading is 090° true (Figure 500, A) and variation is 6°E, the magnetic heading is 090°- 6°= 084° (Figure 500, B). If the deviation on this heading is 2°W, the compass heading is 084°+ 2°= 086° (Figure 500, C). Also, compass error is 6°E-2°W= 4°E, and compass heading is 090°- 4°= 086°. If compass error is easterly, the compass reads too low (in comparison with true directions), and if it is westerly, the reading is too high. Many rules-of-thumb have been devised as an aid to the memory, and any which assist in applying compass errors in the right direction are of value. However, one may forget the rule or its method of application, or may wish to have an independent check. If they understand the explanation given above, they can determine the correct sign without further information. The same rules apply to the use of gyro error. Since variation and deviation are compass errors, the process of removing either from an indication of a direction (converting compass to magnetic or magnetic to true) is often called **correcting**. Conversion in the opposite direction (inserting errors) is then called **uncorrecting**.

Example. - A vessel is on course 215° true in an area where the variation is 7°W. The deviation is as shown in Figure 502. Degaussing is off. The gyro error (GE) is 1° E. A lighthouse bears 306.5° by magnetic compass.

Required.- (1) Magnetic heading (MH).
(2) Deviation.
(3) Compass heading (CH).
(4) Compass error.
(5) Gyro heading.
(6) Magnetic bearing of the lighthouse.
(7) True bearing of the lighthouse.
(8) Relative bearing of the lighthouse.

Solution. -

	TH	215°
	V	7°W
(1)	MH	222°
(2)	D	1.5°W
(3)	CH	223.5°

The deviation is taken from the deviation table (Figure 502) to the nearest half degree.

(4) Compass error is 7° W + 1.5° W = 8.5° W.

	TH	215°
	GE	1°E
(5)	Hpgc	214°
	CB	306.5°
	D	1.5°W
(6)	MB	305°
	V	7°W
(7)	TB	298°

(8) RB='I'B-TH=298°-215°= 083°.

Answers -

(1) MH 222°
(2) D 1.5W°
(3) CH 223.5°
(4) CE 8.5°W
(5) Hpgc 214°
(6) MB 305°
(7) TB 298°
(8) RB 083°

Problem 1 - Fill in the blanks to this table						
	TC	V	MC	D	CC	CE
	°	°	°	°	°	°
(1)	105	15 E	-	5W	-	-
(2)	-	-	-	4 E	215	14 E
(3)	-	12 W	-	-	067	7 W
(4)	156	-	166	-	160	-
(5)	222	-	216	3 W	-	-
(6)	009	-	357	-	-	10 E
(7)	-	2 W	-	6 E	015	-
(8)	-	-	210	-	214	1 W

Answers to Problem 1.- (1) MC 090°, CC 095°, CE 10°E; (2) TC 229°, V 10°E, MC 219°; (3) TC 060°, MC 072°, D 5°E; (4) V 10°W, D 6°E, CE 4°W; (5) V 6°E, CC 219°, CE 3°E; (6) V 12°E, D 2°W, CC 359°; (7) TC 019°, MC 021°, CE 4°E; (8) TC 213°, V 3°E, D 4°W.

Problem 2: A vessel is on course 150° by compass in an area where the variation is 19°E. The deviation is as shown in Figure 502. Degaussing is on.

Required. - (1) Deviation.

(2) Compass error.

(3) Magnetic heading.

(4) True heading.

Answers to Problem 2. - (1) D 1° E, (2) XE 20° E, (3) MH 151°, (4) TH 170°.

Problem 3: A vessel on a course of 055° by gyro and 041° by magnetic compass. The gyro error is 1° W. The variation is 15° E.

Required. - The deviation on this heading.

Answer to Problem 3. - 2° W.

Problem 4: A vessel is on course 177° by gyro. The gyro error is 0.5° E. A beacon bears 088° by magnetic compass in an area where variation is 11° W. The deviation is as shown in Figure 502. degaussing off.

Required. - The true bearing of the beacon.

Answer to Problem 4. - TB 076°.

MAGNETIC COMPASS TABLE NAVSHIPS RPT. 3530-2
NAVSHIPS 3120/4 (REV. 6-67) (FRONT) *(Formerly NAVSHIPS 1104)*
S/N 0105-601-9520

U.S.S. Truckee NO. AO 147 *(BB, CL, DD, etc.)*

[X] PILOT HOUSE [ ] SECONDARY CONNING STATION [ ] OTHER ______

BINNACLE TYPE: [X] NAVY STD [ ] OTHER ______

COMPASS 7 1/2 MAKE Lionel SERIAL NO. 1592

TYPE CC COILS "K" DATE ______

READ INSTRUCTIONS ON BACK BEFORE STARTING ADJUSTMENT

SHIPS HEAD MAGNETIC	DEVIATIONS DG OFF	DEVIATIONS DG ON	SHIPS HEAD MAGNETIC	DEVIATIONS DG OFF	DEVIATIONS DG ON
0	0.5E	0.5E	180	0.5W	0.0
15	1.0E	1.0E	195	1.0W	0.5W
30	1.5E	1.5E	210	1.0W	1.0W
45	2.0E	1.5E	225	1.5W	1.5W
60	2.0E	2.0E	240	2.0W	2.0W
75	2.5E	2.5E	255	2.0W	2.5W
90	2.5E	3.0E	270	1.5W	2.0W
105	2.0E	2.5E	285	1.0W	1.5W
120	1.5E	2.0E	300	1.0W	1.0W
135	1.5E	1.5E	315	0.5W	0.5W
150	1.0E	1.0E	330	0.5W	0.5W
165	0.0	0.5E	345	0.0	0.0

DEVIATIONS DETERMINED BY: [ ] SUN'S AZIMUTH [X] GYRO [ ] SHORE BEARINGS

B 4 MAGNETS RED [ ] FORE [X] AFT AT 14" FROM COMPASS CARD

C 4 MAGNETS RED [ ] PORT [X] STBD AT 10" FROM COMPASS CARD

D 2-7" [X] SPHERES [ ] CYLS AT 12" [X] ATHWARTSHIP [ ] SLEWED ---° [ ] CLOCKWISE [ ] CTR. CLOCKWISE

HEELING MAGNET: [ ] RED UP [X] BLUE UP 10" FROM COMPASS CARD FLINDERS BAR: [X] FORE [ ] AFT 14"

[X] LAT 36°10' N [ ] H [X] LONG 75° 20' W [ ] Z

SIGNED *(Adjuster or Navigator)* T. PARRISH APPROVED *(Commanding)* R. MOSS

Figure 502. Deviation table.

CHAPTER 6

COMPASS ERROR

DETERMINING COMPASS ERROR USING (PUB. NO. 229) SIGHT REDUCTION TABLES FOR MARINE NAVIGATION

600. Compass Error

One of the more frequent applications of sight reduction tables is their use in computing the azimuth of a celestial body for comparison with an observed azimuth in order to determine the error of the compass. In computing the azimuth of a celestial body, for the time and place of observation, it is normally necessary to interpolate the tabular azimuth angle as extracted from the tables for the differences between the table arguments and the actual values of declination, latitude, and local hour angle. The required triple interpolation of the azimuth angle is effected as follows:

1. The main tables are entered with the nearest integral values of declination, latitude, and local hour angle; for these arguments, a base azimuth angle is extracted.
2. The tables are reentered with the same latitude and LHA arguments but with the declination argument 1° greater or less than the base declination argument depending upon whether the actual declination is greater or less than the base argument. The difference between the respondent azimuth angle and the base azimuth angle establishes the azimuth angle difference (Z Diff.) for the increment of declination.
3. The tables are reentered with the base declination and LHA arguments but with the latitude argument 1° greater or less than the base latitude argument depending upon whether the actual (usually DR) latitude is greater or less than the base argument to find the Z Diff. for the increment of latitude.
4. The tables are reentered with the base declination and latitude arguments, but with the LHA argument 1° greater or less than the base LHA argument depending upon whether the actual LHA is greater or less than the base argument to find the Z Diff. for the increment of LHA.
5. The correction to the base azimuth angle for each increment is $\text{Z Diff.}\ \frac{\text{Inc.}}{60'}$.

Example.-In DR Lat. 13°24.0'N, the azimuth of the Sun is observed as 070.3° pgc. At the time of the observation, the declination of the Sun is 20°13.8'N; the local hour angle of the Sun is 276°41.2'. The error of the gyrocompass is found as follows:

Actual		Base Arguments	Base Z	Tab* Z	Z Diff	Increments	Correction (Z Diff Inc. 60)
Dec.	20°13.8′N	20°	71.8°	70.8°	–1.0°	13.8′	–0.2°
DR Lat.	13°24.0N	13° (Same)	71.8°	71.9°	+0.1°	24.0′	0.0°
LHA	276°41.2′	277°	71.8°	71.6°	–0.2°	18.8′	–0.1°
Base Z	71.8°					Total Corr.	-0.3°
Corr.	(-) 0.3°						
Z	N71.5°E						
Zn	071.5°						
Zn pgc	070.3°						
Gyro Error	1.2°E						

* Respondent for two base arguments and1° change from third base argument, in vertical order of Dec., DR Lat., and LHA.