CHAPTER 3

NAVIGATIONAL ERRORS

DEFINING NAVIGATIONAL ERRORS

300. Introduction

As commonly practiced, navigation is not an exact science. A number of approximations which would be unacceptable in careful scientific work are used by the navigator, because greater accuracy may not be consistent with the requirements or time available, or because there is no alternative.

Thus, when the navigator uses his latitude graduations as a mile scale or computes a great-circle course and distance, s/he neglects the flattening of the earth at the poles, a practice that is not acceptable to the geodetic surveyor. When the navigator plots a visual bearing or an azimuth line for a celestial line of position, s/he uses a rhumb line to represent a great circle on a Mercator chart. When s/he plots the celestial line of position, s/he substitutes a rhumb line for a small circle. When the navigator interpolates in sight reduction or lattice tables, s/he assumes a linear (constant-rate) change between tabulated values. When s/he measures distance by radar or depth by echo sounder, s/he assumes that the radio- or sound-wave has constant speed under all conditions. When the navigator applies dip and refraction corrections to his or her sextant altitude, s/he generally assumes standard atmospheric conditions. These are only a few of the approximations commonly applied by a navigator.

There are so many that there is a natural tendency for some of them to cancel others. Thus, under favorable conditions, a position at sea determined from celestial observation by an experienced observer should seldom be in error by more than 2 miles. However, if the various small errors in a particular observation all have the same sign (all plus or all minus), the error might be several times this amount without any mistake having been made by the navigator.

Greater accuracy could be attained, but at a price. The navigator is a practical individual. In the course of ordinary navigation, s/he would rather spend 10 minutes determining a position having a probable error of plus or minus 2 miles, than to spend several hours learning where s/he was to an accuracy of a few meters. But if the navigator can determine a recent or present position to greater accuracy, the decrease in error is attractive. The various navigational aids have been designed with this in mind. Greater accuracy in plotting could be achieved by increasing the scale of the chart or plotting sheet. This has been done for confined waters where a higher degree of accuracy is needed, but a large scale plotting sheet would be a nuisance at sea. The hand-held marine sextant is not sufficiently accurate for use in determining an astronomical position in a geodetic survey. But, it is much more satisfactory at sea than the surveyor's astrolabe or theodolite, which require stable platforms if their potential accuracy is to be realized.

An understanding of the kinds of errors involved in navigation, and of the elementary principles of probability, should be of assistance to a navigator in interpreting his or her results.

301. Definitions

The following definitions apply to the discussions of this chapter:

Error is the difference between a specific value and the correct or standard value. As used here it does not include mistakes, but is related to lack of perfection. Thus, an altitude determined by marine sextant is corrected for a standard amount of refraction, but if the actual refraction at the time of observation varies from the standard, the value taken from the table is in error by the difference between standard and actual refraction. This error will be compounded with others in the observed altitude. Similarly, depth determined by echo sounder is in error, among other things, by the difference between the actual speed of sound waves in the water and the speed used for calibration of the instrument. The depth will also be in error if an echo is returned from a phantom bottom instead of from the actual bottom. This chapter is concerned primarily with the deviation from standards. Thus, while variation of the compass is an error when referred to true directions, the difference between the assumed variation and that actually existing is an error with reference to magnetic direction. Corrections can be applied for standard values of error. It is the deviation from standard, as well as mistakes, that produce inaccurate results in navigation. Various kinds of errors are discussed in the following articles.

Mistake is a blunder, such as an incorrect reading of an instrument, the taking of a wrong value from a table, or the plotting of a reciprocal bearing. The mistake is discussed in more detail in Section 312.

Standard is something established by custom, agreement, or authority as a basis for comparison. It is customary to use nautical miles for measuring distances between ports. By international agreement the nautical mile is defined as exactly 1852 meters. By authority of various countries which are parties to the agreement, this length is translated to the linear units adopted by that country. It is the fact of establishment or general acceptance that determines whether a given quantity or condition has become a standard of measure or quality.

Thus, in 1960, the standard unit of length agreed upon at the Eleventh General (International) Conference on Weights and Measures to redefine the meter was 1,650,763.73 wavelengths of the orange-red radiation in vacuum of krypton 86 corresponding to the unperturbed transition between the 2p10 and 5d5 levels. This established standard of length now serves as a basis for measurement of any physical magnitude, as the length of the meridian. Multiples and submultiples of a standard are exact. In 1959, the U.S. adopted the exact relationships of 1 yard as equal to 0.9144 meter and 1 inch as equal to 2.54 centimeters. Hence, 39.37 U.S. inches are approximately equal to 1 meter. Because 1 foot equals 12 inches by definition, and the international nautical mile has been defined as 1852 meters, the international nautical mile is equal to 6,076.11549 U.S. feet (approximately). The previous U.S. foot (6,076.10333 . feet equals 1 nautical mile) has been re-designated as the U.S. survey foot.

Frequently, a standard is chosen so that it serves as a model which approximates a mean or average condition. However, the distinction between the standard value and the actual value at any time should not be forgotten. Thus, a standard atmosphere has been established in which the temperature, pressure, density, etc., are precisely specified for each altitude. Actual conditions, however, are generally different from those defined by the standard atmosphere. Similarly, the values for dip given in the almanacs are considered standard by those who use them, but actual dip may be appreciably different from that tabulated.

Accuracy is the degree of conformance with the correct value, while **precision** is the degree of refinement of a value. Thus, an altitude determined by a marine sextant might be stated to the nearest 0.1', and yet be accurate only to the nearest 1.0' if the horizon is indistinct.

302. Systematic Errors

Systematic errors are those which follow some law by which they can be predicted. The accuracy with which a systematic error can be predicted depends upon the accuracy with which the governing law is understood. An error which can be predicted can be eliminated, or compensation can be made for it.

The simplest form of systematic error is one of unchanging magnitude and sign. This is called a **constant error**. Examples are the index error of a marine sextant, watch error, or the error resulting from a lubber's line not being accurately aligned with the longitudinal axis of the craft. In each of these cases, all readings are in error by a constant amount as long *as the adjustment remains unchanged*, and can be removed by applying a correction of equal magnitude and opposite sign. Index error and watch error can be removed by adjustment of the instrument. Lubber's line error can be removed by aligning the lubber's line with the longitudinal axis of the craft.

Another type of systematic error results from a nonstandard rate. If a watch is gaining 4 seconds per day, its readings will be in error by 1 second after an interval of 6 hours, 8 seconds at the end of 2 days, etc. This principle is used in establishing a chronometer rate (Section 1608, Volume 1, 2019 edition) for determination of chronometer error between comparisons of the chronometer with time signals. It can be eliminated by adjusting the rate. If a current is running and no allowance for it is made in the dead reckoning, the DR position is in error by an amount proportional to elapsed time. The error introduced by maintaining heading by means of an inaccurate compass is proportional to distance, as is the lateral error in a line of position plotted from an inaccurate bearing.

One of the causes of equation of time (Section 1601, Volume 1, 2019 edition) is the fact that the ecliptic, around which annual motion occurs, is not parallel to the celestial equator, around or parallel to which apparent daily motion takes place. The same type of systematic error is involved in other measurements. Consider the measurement of bearing with a tilted compass card. Bearing is measured by a system of uniform graduations (degrees) of a circle (such as a compass card) in the horizontal plane. If the card is tilted, and its graduations are projected onto the horizontal plane, the circle becomes an ellipse with the graduations unequally spaced. Along the axis of tilt and a line perpendicular to it, directions are correct. But near the axis of tilt the graduations are too close together, and near the perpendicular they are too widely spaced.

The error thus introduced is similar to that which would arise if a watch face were tilted but the motion of the hands remained horizontal. If it were tilted around the "3-9" line, it would appear to run slow near the hour and half hour, and fast near the quarter and three-quarter hours. If the direction to be observed is of an object above or below the horizontal, as the azimuth of a celestial body, measurement is made to the foot of the perpendicular through the object.

The sight vanes of a compass move in a plane perpendicular to the compass card. Hence, if the card is tilted, measurement is made to the foot of a perpendicular to the card, rather than to the foot of a perpendicular to the horizontal, introducing an error which increases with the angle of tilt and also with the angle of elevation (or depression) of the object. This error is greatest along the axis of tilt, and zero along the perpendicular to it. Both of these tilt errors can be corrected by leveling the compass card.

A different type of tilt error occurs when a reflection takes place from a tilted surface, such as the ionosphere, the error being proportional to the angle of tilt. In some respects, this error is similar to coastal refraction of a radio wave.

Additional examples of systematic error are uncorrected deviation of the compass, error due to a position in a pattern of hyperbolas, error due to incorrect location of a Loran transmitter, uncorrected parallax, and uncorrected personal error.

303. Random Errors

Random errors are chance errors, unpredictable in magnitude or sign. They are governed by the laws of probability. If the altitude of a celestial body is observed, the reading may be (1) too great, (2) correct, or (3) too small. If a number of observations are made, and there is no systematic error, the probability of a positive error is exactly equal to the probability of a negative error. This does not mean that every second observation having an error will be too great. However, the greater the number of observations, the greater is the probability that the percentage of positive errors will equal the percentage of negative ones, and that their magnitudes will correspond.

Error	No. of obs.	Percent of obs.
- 10′	0	0. 0
- 9′	1	0. 2
- 8′	2	0. 4
- 7′	4	0. 8
- 6′	9	1. 8
- 5′	17	3. 4
- 4′	28	5. 6
- 3′	40	8. 0
- 2′	53	10. 6
- 1′	63	12. 6
0	66	13. 2
+ 1′	63	12. 6
+ 2′	53	10. 6
+ 3′	40	8. 0
+ 4′	28	5. 6
+ 5′	17	3. 4
+ 6′	9	1. 8
+ 7′	4	0. 8
+ 8′	2	0. 4
+ 9′	1	0. 2
+10′	0	0. 0
0	500	100. 0

Table 303. Normal distribution of random errors.

Suppose that 500 observations are made, with the results shown in Table 303. A close approximation of the plot of these errors is shown in Figure 303a. The plot has been modified slightly to constitute the normal curve of random errors, which is the same as the actual curve except that the normal curve approaches zero as the error increases, while the actual curve reaches zero at (+)10' and (-)10'. The height of the curve at any point represents the percentage of obser-

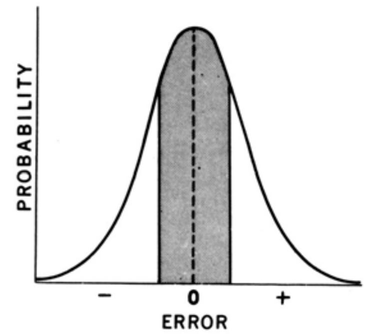

Figure 303a. Normal curve of random error with 50 percent of area shaded. Limits of shaded area indicate probable error.

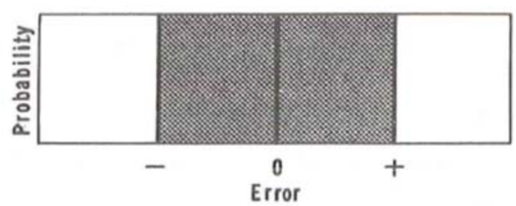

Figure 303b. Rectangular error, with 50 percent area shaded.

vations that can be expected to have the error indicated at that point. The probability of any similar observation having any given error is the proportion of the number of observations having this error to the total number of observations, or the percentage expressed as a decimal. Thus, the probability of an observation having an error of -3' is

$$\frac{40}{500} = \frac{1}{12.5} = 0.08(8\%)$$

If the area under the curve represents 100 percent of the observations, half the area (the shaded portion of Figure 303c) represents 50 percent of the observations. The value of the error at the limits of this shaded portion is often called the “50 percent error,” or **probable error**, meaning that 50 percent of the observations can be expected to have less error, and 50 percent greater error. Similarly, the limits which contain the central 95 percent of the area denote the 95 percent error. The percentage of error is found mathematically. For a normal curve, each error is squared, the sum of the squares is divided by one less than the number of observations, and the square root of the quotient is determined. This value is called the **standard deviation** or **standard error** (σ , the Greek letter sigma). In the illustration, the standard deviation is the square root of:

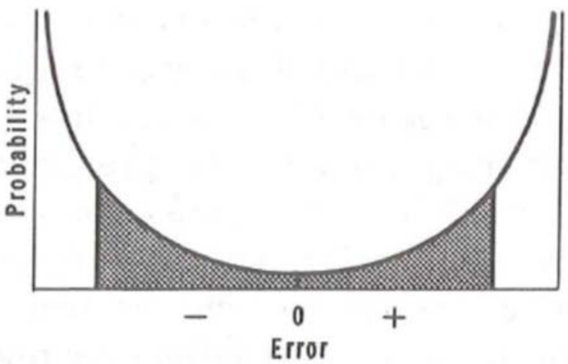

Figure 303c. Periodic error, with 50 percent area shaded.

$$0 \quad (-10)^2 + 1 \quad (-9)^2 + 2 \quad (-8)^2 + 4 \quad (-7)^2 + 9 \quad (-6)^2, \text{etc}$$

divided by 499 or

$$\frac{4474}{\sqrt{499}} = \sqrt{8.966} = 2.99 \text{ (about 3)}$$

The standard deviation is the 68.27 percent error. The probability of the occurrence of an error of or less than a specific magnitude may be approximately determined by the following relationship (with the answers for the illustration given):

50% error = 2/3 x σ = 2' (approx.)

68% error = 1 x σ = 3' (approx.)

95% error = 2 x σ = 6' (approx.)

99% error = 2 2/3 x σ = 8' (approx.)

99.9% error =3 1/3 x σ = 10' (approx.)

Many of the errors of navigation do not follow the normal distribution discussed above. *Pub. No. 229* values of altitude can be taken only to the nearest 0.1'. The error in tabular altitude might have any value from (+) 0. 05' to (-) 0.05', and any value within these limits is as likely to occur as any other of the same precision. The same is true of a sextant that cannot be read more precisely than 0.1', and of a time-difference that cannot be measured more precisely than 1 µs. These values refer to the single errors indicated, and not to the total error that might be involved. This is a **rectangular error**, so called because of the shape of its plot, as shown in Figure 303b. The 100 percent error is half the difference between readings. The 50 percent error is half this amount, the 95 percent error is 0.95 times this amount, etc. In some cases it may be more meaningful to refer to the rectangular error as the **resolution error**.

Still another type random error is encountered in navigation. If a compass is fluctuating periodically due to yaw of a ship, its motion slows as the end of a swing is approached, when the error approaches maximum value. If readings were taken continuously or at equal intervals of time, the interval being a small percentage of the total period of oscillation, the curve of errors would have a characteristic U-shape, as shown in Figure 303c. The same type error is involved in measurement of altitude of a celestial body from a wing of the bridge of a heavily rolling vessel, when the roll causes large changes in the height of eye. This type of error is called a **periodic error**. The effect is accentuated by the tendency of the observer to make readings near one of the extreme values because the instrument appears steadiest at this time. If it is impractical to make a reading at the center of the period, the error can be eliminated or reduced by averaging readings taken continuously or at short intervals, as indicated above. This is the method used in averaging type artificial-horizon sextants. Generally, better results can be obtained by taking maximum positive and maximum negative readings, and averaging the results.

The curve of any type of random error is symmetrical about the line representing zero error. This means that in the ideal plot every point on one side of the curve is error of the same magnitude. The average of all readings, considering signs, is zero. The larger the number of readings made, the greater the probability of the errors fitting the ideal curve. Another way of stating this is that as the number of readings increases, the error of the average can be expected to decrease

304. Combinations of Errors

Many of the results obtained in navigation are subject to more than one error. Chapter 19, Volume 1, lists 19 errors applicable to sextant altitudes. Some of these have several components. A number of possible errors are involved in the determination of computed altitude and azimuth. A rectangular error is possible in finding the altitude difference. Several additional errors may affect the accuracy of plotting. Thus, the line of position as finally plotted may include 30 errors or more. Corrections are applied for some of the larger ones, so that in each of these cases the applicable error is the difference between the applied correction and the actual error. Thus, a dip correction may be applied for a height of eye of 30 feet, while the actual height at the moment of observation may be 31 feet 6 inches. Even if the height of eye is exactly 30 feet, a rectangular error may be involved in taking the dip correction from the table

If two or more errors are applicable to a given result, the total error is equal to the algebraic sums of all errors. Thus, if a given number is subject to errors of (+) 4, (-) 2, (-) 1, (+) 3, (+) 2, 0, and (-) 2, the total error is (+) 4. Systematic errors can be combined by adding the curves of

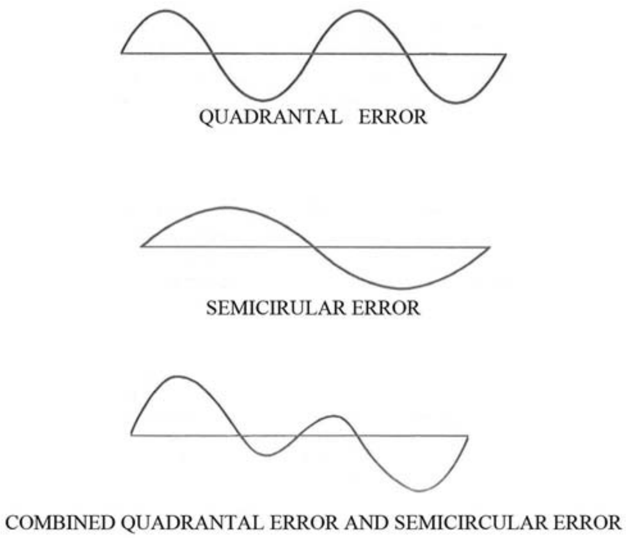

Figure 304. Combining systemic error.

individual errors. Thus, a magnetic compass may have a quadrantal error as shown by the top curve of Figure 304, and a semicircular error as shown by the second curve. The sum of these two errors is shown in the bottom curve. If, in addition, the compass has a constant error, the bottom curve is moved vertically upward or downward by the amount of the constant error, without undergoing a change of form. If the constant error is greater than the maximum value of the combined curves, all errors are positive or all are negative, but of varying magnitude.

If a number of random errors are combined, the result tends to follow a normal curve regardless of the shape of the individual errors, and the greater the number, the more nearly the result can be expected to approach the normal curve (Figure 303a). If a given result is subject to errors of plus or minus 3, 2, 1, 2, 4, 2, 1, 8, 1, and 2, the total error could be as much as 26 if all errors had the same sign. However, if these are truly random, the probability of them all having the same sign is only 1 in 1024. This is so because the chance of any one being positive (or negative) is one half. By the same reasoning, approximately half of the positive (or negative) results will have any one particular additional correction positive (or negative). Thus, the probability of any two particular corrections having a positive (or negative) sign is $1/2 \quad 1/2 = (1/2)^2 = \frac{1}{4}$. The probability of all 10 corrections having a positive (or negative) sign is $(1/2)^{10} = \frac{1}{1024}$. If there were 20 corrections, the probability of all having a positive (or negative) sign would be $(1/2)^{20} = \frac{1}{1048576}$.

When both systematic and random errors are present in a process, both effects are present. An increase in the number of readings decreases the residual random error, but regardless of the number of readings, a systematic error is present in its entirety. Thus, if a number of phase-difference readings are made at a fixed point, the average should be a good approximation of the true value if there is no systematic error. But if the equipment is out of adjustment to the extent that the lane is incorrectly identified, no number of readings will correct this error. In this illustration, a constant error is combined with a normal random error. The normal curve has the correct shape, but is offset from the zero value.

Under some conditions, systematic errors can be eliminated from the results even when the magnitude is not determined. Thus, if two celestial bodies differ in azimuth by 180°, and the altitude of each is observed, the line midway between the lines of position resulting from these observations is free from any *constant* error in the *altitude* (such as abnormal refraction or dip, or incorrect IC). It would *not* be free from such a constant error as one in time (unless the bodies were on the celestial meridian). Similarly, a fix obtained by observations of three stars differing in azimuth by 120°, or four stars differing by 90° is free from constant error in the altitude, if the center of the figure made by the lines of position is used. The center of the figure formed by circles of position from distances of objects equally spaced in azimuth is free from a constant error in range. A constant error in bearing lines does not introduce an error in the fix if the objects are equally spaced in azimuth. In all of these examples, the correct position is *outside* the figure formed by the lines of position if all objects observed are on the same side of the observer (that is, if they lie within an arc of less than 180°).

305. Navigation Accuracy

Navigation accuracy is normally expressed in terms of the probability of being within a specified distance of a desired point during the navigation process.

If the accuracy of only a single line of position is being considered, the specified distance may be stated as the standard deviation (Section 303) or some multiple thereof, assuming that the errors of the line of position follow a **single-axis normal distribution.** The distance as stated for the standard deviation of a line of position is measured from the arithmetic mean of the positions which could be established from a large number of observations at a given place and time. Therefore, this distance does not indicate the separation between the line of position and the observer's actual position, except by chance. If the error is stated as 1 σ, 68.27 percent of the cases should result in line of position displacements from the arithmetic mean in any direction not exceeding the distance specified for 1 σ. If the error is stated as 2 σ, 95.45 percent of the lines of position should not be displaced from the arithmetic mean in any direction by more than the distance specified for 2 σ. If the error is stated as the probable error, 50 percent of the lines

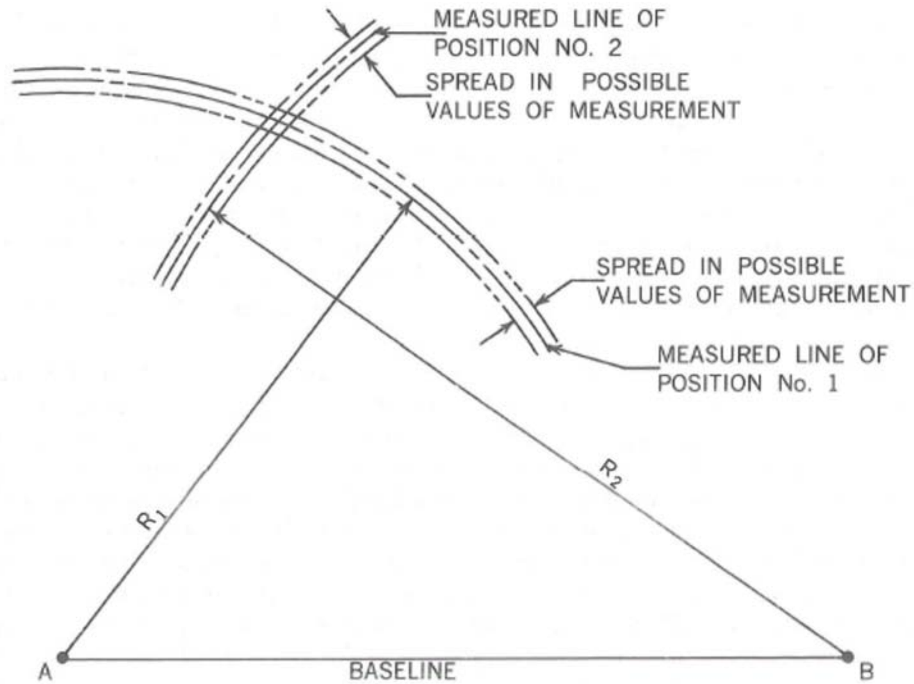

Figure 305a. Fix established at intersection of two lines of position having different values of error.

of position should not be displaced from the arithmetic mean in any direction by more than the distance specified for 0.6745 σ.

The standard deviation is also employed in developing expressions for the probability of a fix position being within a specified distance of the mean of the positions which could be established from a large number of observations at a given place and time by means of the system used to establish the fix.

In the following discussion, the fix is established by the intersection of two lines of position, each of which may be in error. The lines of position (Figure 305a) are range measurements from two points at the extremities of a baseline of known length. Because of inaccuracies in measurement, the actual ranges differ from the measured values and may lie somewhere between the limits which are shown as additional arcs either side of the measured arc.

The intersection of the two lines of position together with the standard deviations associated with each is drawn to an expanded scale in Figure 305b. It can be shown that the *contours of equal probability density* about such an intersection are ellipses with their center at the intersection. Thus, the ellipse shown in Figure 305b might be the 75 percent probability ellipse, meaning that there are three chances in four that a fix will lie within such ellipse centered upon the mean of the positions which would be established from a large number of observations at a given place and time by means of the system used to establish the fix.

For simplicity in this discussion of navigation accuracy, the following assumptions are made:

1. All constant errors or **bias errors** have been removed, leaving only the random errors. Thus, the mean or average error is assumed to be zero.
2. These random errors are assumed to be normally distributed.
3. The errors associated with the two intersecting lines of position are assumed to be independent. This assumption implies that a change in the error of one line of position has no effect upon the other.
4. The lines of position are assumed to be straight lines in the small area in the immediate vicinity of their intersection. This assumption is valid so long as the standard deviation is small compared to the radius of curvature of the line of position.
5. Errors of position are limited to the two-dimensional case. As shown in Figure 305b, the general case of the intersection of two lines of position at any angle of cut and with different values of error associated with each line of position results in an elliptical error figure. Figure 305c shows the ellipse simplified to geometrical terms.

One may readily surmise from Figure 305c that the exact shape of the error figure varies with the magnitudes of the two one-dimensional input errors, $\sigma 1$ and $\sigma 2$ as well as with the angle of cut, α. The angle α is also the angle between the two values of sigma because the standard

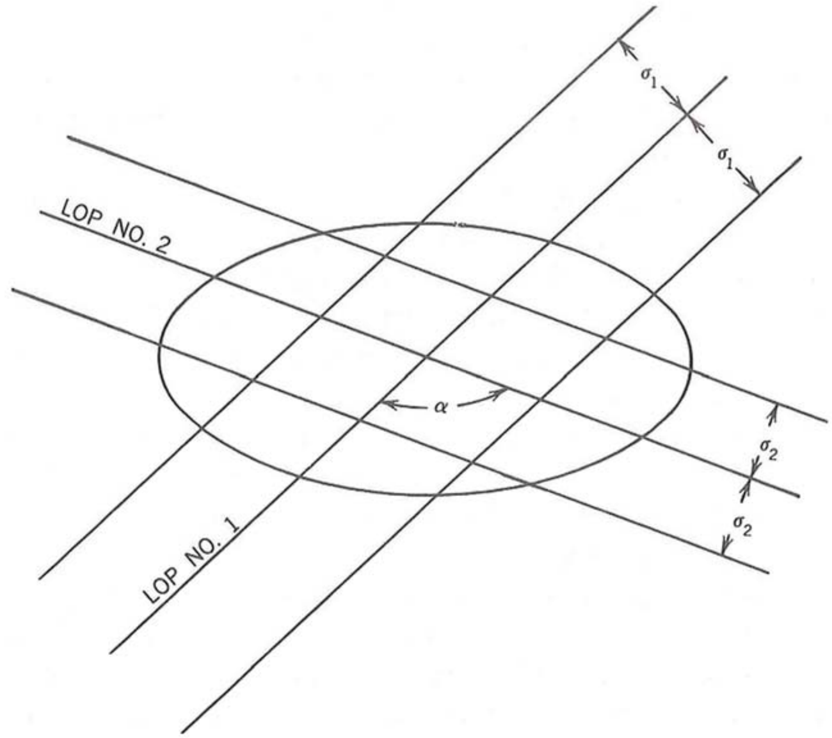

Figure 305b. Expanded view of intersection of two lines of position.

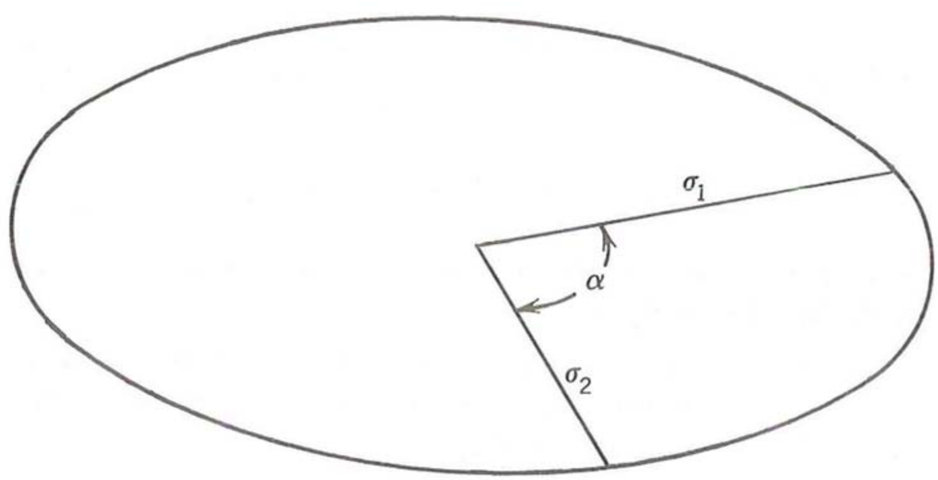

Figure 305c. Basic error ellipse.

deviations are mutually perpendicular to their corresponding lines of position. These variations can be calculated to provide the probability that a point is located within a circle of stated radius.

When this is done, the error is stated in terms more meaningful to the practicing navigator. The basis of this concept may best be seen by first considering the special case when the two errors are equal, and the angle of intersection of the lines of position is a right angle. In this case, *and in this case alone*, the error figure becomes a circle and is described by the circular normal distribution. A plot of this special function is given in Figure 305d. In this plot, the horizontal axis is measured in terms of R/σ, R being the stated radius of the circle and σ being the measure of error. The error measure is given simply as σ, for in this circular case $\sigma 1 = \sigma 2$. To illustrate, a measurement system gives a circular error figure and has a value of $\sigma = 100$ meters; the probability of actually being located within a circle of 100 meters radius when $R/\sigma = 1.0$ may be read from the verti-

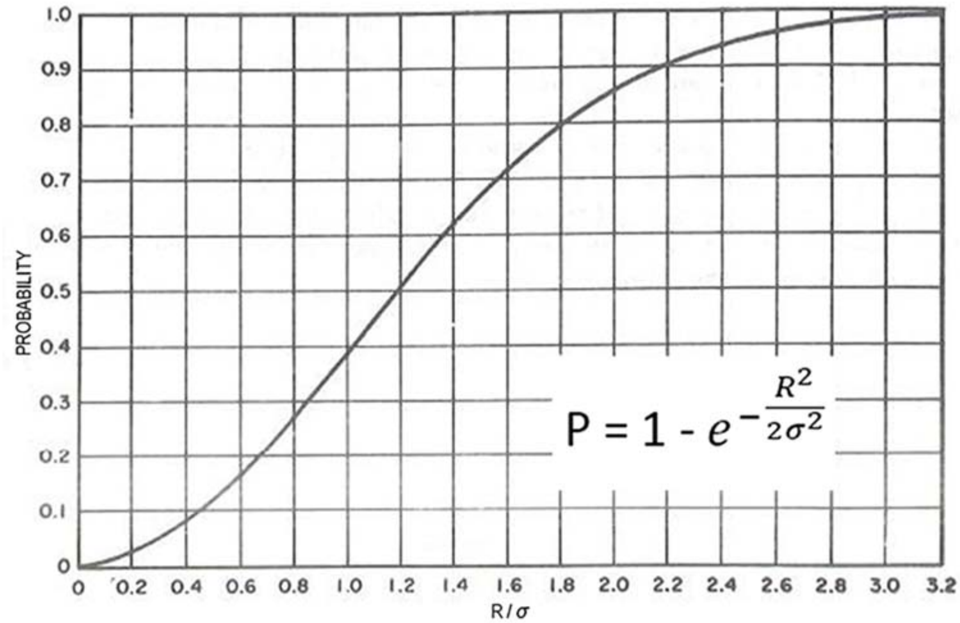

Figure 305d. Circular normal distribution.

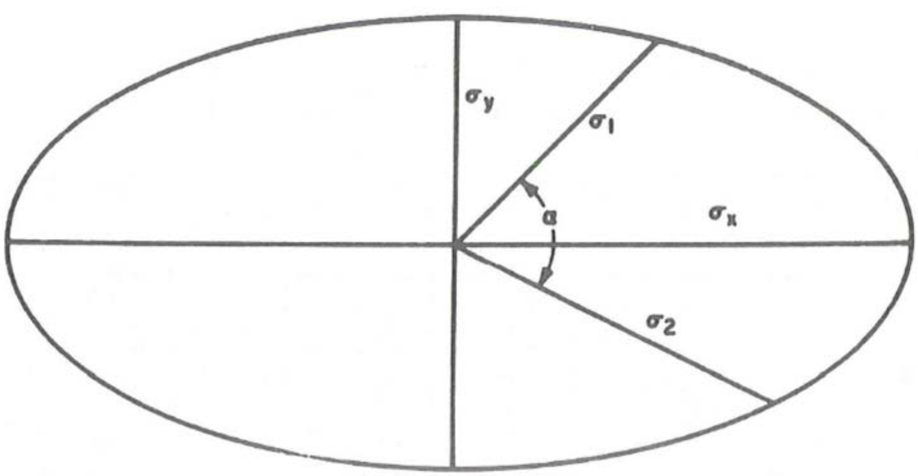

Figure 305e. Transformation to standard deviations along ellipse axes.

cal axis to be 39.3 percent. To obtain the radius of a circle within which a 50 percent probability results, the corresponding value of R/σ is seen to be 1.18 from the graph. Thus, for this example, the **circular probable error** (**CPE** or **CEP** or circle of 50% probability) would be 118 meters..

In one method of using error ellipses to obtain the radii of **circles of equivalent probability**, new values of σ are found along the major and minor axes of the ellipse (Figure 305e) using the following equations:

$$\sigma x^2 = \frac{1}{2\sin 2\alpha}\left[\sigma 1^2 + \sigma 2^2 + \sqrt{(\sigma 1^2 + \sigma 2^2)^2 - 4\sin 2\alpha\sigma_1^2\sigma_2^2}\right]$$

$$\sigma y^2 = \frac{1}{2\sin 2\alpha}\left[\sigma 1^2 + \sigma 2^2 - \sqrt{(\sigma 1^2 + \sigma 2^2)^2 - 4\sin 2\alpha\sigma_1^2\sigma_2^2}\right].$$

Then the ratio $c = \frac{\sigma_y}{\sigma_z}$ where σ_x is the larger of the two new standard deviations, is used in entering Table 305a which relates ellipses of varying values of ellipticity to the radii of circles of equivalent probability.

For a numerical example to illustrate the method of calculation, assume that the angle of cut α is 50°, σ 1 is 15 meters, and σ 2 is 20 meters to determine the probability of location within a circle of 30 meters radius.

For the computation the following numbers are needed:

$$\sigma_1^2 = 225$$

$$\sigma_2^2 = 400$$

$$\sin 2\sigma = 0.5868$$

Substituting in the equations for σ_x^2 and σ_y^2, σ_x and σ_y are calculated as 29.9 meters and 13.1 meters, respectively. Since the function *K* multiplied by the larger of the two standard deviations obtained by the transformation method gives the value of the radius of the circle of the corresponding value of probability shown in Table 305a, *K*=1.003. On entering Table 305a with *K*=1.0 and c= 0.44, the probability is found to be 62 percent.

Table 305b and Figure 305g provide ready information about the sizes of circles of specific probability value associated with ellipses of varying eccentricities.

In another method, fictitious values of sigma of identical value, indicated by $\sigma *$, are assumed to replace the two unequal values originally given (σ 1and σ 2). A fictitious angle of cut $\alpha *$ is also assumed to replace the angle of cut (α) originally given (Figure 305f).

The method utilizes a set of probability curves, with a separate curve for each value of angle of cut (Figure 305h). These curves can be used only when the two error measures are equal, hence the need for making the transformation to the fictitious $\sigma *$.

The values of $\sigma *$ and $\alpha *$ needed to utilize the probability curves may either be determined from Figure 305j and Figure 305i or by means of the following equations:

$$\sigma* = \frac{\sin\beta\sqrt{\sigma_1^2 + \sigma_2^2}}{\sqrt{2}}$$

$$\alpha* = arc\sin(\sin 2\beta \sin\alpha)$$

where

$$\beta = arc\tan(\sigma_1/\sigma_2)$$

Thus,

$$\sin 2\beta = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}.$$

To use the curve and nomogram for obtaining $\sigma *$ and $\alpha *$, one must first calculate the ratio σ_2/σ_1. The value σ_1, is always taken as the larger of the two in the ratio so that the ratio is always less than 1.0. With this ratio, enter the curve of Figure 305j and obtain the $\sigma *$factor. Multiply σ_1 by this factor to obtain the fictitious function $\sigma *$. The nomogram of Figure 305i is entered with the same ratio to obtain the fictitious angle of cut $\alpha *$.

For a numerical example to illustrate the method of calculation, assume that the angle of cut of 50°, σ_1, is 20 meters, and σ_2 is 15 meters to determine the probability of location within a circle of 30 meters radius.

Calculate the ratio $\sigma_2/\sigma_1 = \frac{15}{20} = 0.75$.

Enter the curve of Figure 305j with this ratio and obtain the $\sigma *$ factor (0.845). Multiply this factor by σ_1 to obtain $\sigma *$ equals 16.9 meters. Calculate the ratio

$$R/\sigma* = 30/16.9 = 1.78.$$

Enter the nomogram of Figure 305i with the ratio σ_2/σ_1, and with the given angle α to obtain the fictitious angle of cut $\alpha * = 47°$.

The values $R/\sigma* = 1.78$ and $\alpha* = 47°$ are then used to enter the probability curves of to obtain P= 0.62 or 62 percent, interpolating between the 40° and 50° curves for $\alpha* = 47°$.

GEOMETRIC ERROR CONSIDERATIONS

306. Geometric Error Considerations

From the information that can be derived using the two methods of transformation of elliptical error data, one can develop curves which show for constant values of initial error that the size of a circle of fixed value of probability varies as a function of the angle of cut of the lines of position.

To simplify the investigation of geometrical factors, it is initially desirable to consider the special case of $\sigma_1 = \sigma_2 = \sigma$. Under this special condition, the long equations for σ_x and σ_y can be simplified to facilitate computation as follows:

$$\sigma_x = \frac{\sqrt{2}}{2\sin\frac{1}{2}\alpha}\sigma \qquad (\sigma_1 = \sigma_2)$$

$$\sigma_y = \frac{\sqrt{2}}{2\cos\frac{1}{2}\alpha}\sigma \qquad (\sigma_1 = \sigma_2)$$

Taking the ratio of these two values, a simple equation is found for the ratio *c*

$$c = \frac{\sigma_y}{\sigma_x} = \tan\frac{1}{2}\alpha$$

K \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.0796557	.0443987	.0242119	.0164176	.0123875	.0099377	.0082940	.0071157	.0062299	.0055400	.0049875
0.2	.1585194	.1339783	.0884533	.0628396	.0482413	.0390193	.0327123	.0281415	.0246824	.0219757	.0198013
0.3	.2358228	.2213804	.1739300	.1318281	.1039193	.0851535	.0719102	.0621386	.0546598	.0487639	.0440025
0.4	.3108435	.3010228	.2635181	.2139084	.1742045	.1451808	.1237982	.1076237	.0950495	.0850326	.0768837
0.5	.3829249	.3755884	.3481790	.3003001	.2532953	.2152886	.1857448	.1626829	.1443941	.1296286	.1175031
0.6	.4514938	.4457708	.4255605	.3846374	.3357384	.2914682	.2548177	.2251114	.2009797	.1811783	.1647298
0.7	.5160727	.5115048	.4960683	.4633258	.4170862	.3699305	.3280302	.2925654	.2629373	.2381583	.2172955
0.8	.5762892	.5725957	.5604457	.5349387	.4941882	.4474207	.4025628	.3627122	.3283453	.2989700	.2738510
0.9	.6318797	.6288721	.6191354	.5993140	.5651564	.5213998	.4759375	.4333628	.3953279	.3620135	.3330232
1.0	.6826895	.6802325	.6723586	.6568242	.6291249	.5900953	.5461319	.5025790	.4621421	.4257553	.3934693
1.1	.7286679	.7266597	.7202682	.7079681	.6859367	.6524489	.6116316	.5687467	.5272462	.4887873	.4539256
1.2	.7698607	.7682215	.7630305	.7532175	.7359558	.7079973	.6714269	.6306168	.5893494	.5498736	.5132477
1.3	.8063990	.8050648	.8008554	.7929968	.7793550	.7567265	.7249673	.6873122	.6474394	.6079822	.5704426
1.4	.8384867	.8374049	.8340018	.8277048	.8169851	.7989288	.7720889	.7383089	.7007900	.6623035	.6216889
1.5	.8663856	.8655127	.8627728	.8577362	.8493071	.8350816	.8129287	.7833962	.7489500	.7122546	.6753475
1.6	.8904014	.8897008	.8875060	.8834914	.8768644	.8657559	.8478393	.8226246	.7917194	.7574708	.7219627
1.7	.9108691	.9103102	.9085619	.9053766	.9001746	.8915536	.8773116	.8562471	.8291137	.7977882	.7462539
1.8	.9281394	.9276964	.9263125	.9237989	.9197275	.9130680	.9019110	.8846624	.8613238	.8332175	.8021013
1.9	.9425669	.9422182	.9411299	.9391586	.9359855	.9308615	.9222277	.9083609	.8886731	.8639149	.8355255
2.0	.9544997	.9542272	.9533775	.9518415	.9493815	.9454546	.9388418	.9278799	.9115762	.8901495	.8646647
2.1	.9642712	.9640598	.9634011	.9622127	.9603170	.9573205	.9522999	.9437668	.9305013	.9122714	.8897495
2.2	.9721931	.9720304	.9715237	.9706109	.9691597	.9668845	.9631017	.9565522	.9459386	.9306821	.9110784
2.3	.9785518	.9784275	.9780408	.9773450	.9762419	.9745239	.9716934	.9667306	.9583739	.9458085	.9289946
2.4	.9836049	.9835108	.9832180	.9826918	.9818594	.9805703	.9784661	.9747495	.9682698	.9580804	.9438652
2.5	.9875807	.9875100	.9872900	.9868953	.9862720	.9853112	.9837569	.9810035	.9760522	.9679136	.9560631
2.6	.9906776	.9906249	.9904612	.9901674	.9897045	.9889934	.9878527	.9858331	.9821023	.9756969	.9659525
2.7	.9930661	.9930271	.9929062	.9926894	.9923483	.9918260	.9909944	.9895268	.9867530	.9817837	.9738786
2.8	.9948897	.9948612	.9947727	.9946141	.9943649	.9939842	.9933821	.9923249	.9902888	.9864876	.9801589
2.9	.9962684	.9962477	.9961834	.9960684	.9958878	.9956126	.9951798	.9944246	.9929482	.9900803	.9850792
3.0	.9973002	.9972853	.9972391	.9971564	.9970266	.9968294	.9965205	.9959854	.9949274	.9927925	.9888910
3.1	.9980648	.9980542	.9980212	.9979622	.9978699	.9977296	.9975109	.9971348	.9963851	.9948168	.9918113
3.2	.9986257	.9986182	.9985949	.9985533	.9984880	.9983892	.9982356	.9979733	.9974478	.9963105	.9940240
3.3	.9990332	.9990279	.9990116	.9989824	.9989368	.9988677	.9987607	.9985792	.9982147	.9974004	.9956822
3.4	.9993261	.9993225	.9993112	.9992909	.9992593	.9992115	.9991376	.9990129	.9987626	.9981868	.9969113
3.5	.9995347	.9995323	.9995245	.9995105	.9994888	.9994559	.9994053	.9993204	.9991502	.9987480	.9978125
3.6	.9996818	.9996801	.9996748	.9996653	.9996505	.9996281	.9995938	.9995364	.9994218	.9991442	.9984662
3.7	.9997844	.9997832	.9997797	.9997733	.9997633	.9997482	.9997251	.9996867	.9996102	.9994208	.9989352
3.8	.9998553	.9998545	.9998522	.9998478	.9998412	.9998311	.9998157	.9997902	.9997396	.9996119	.9992682
3.9	.9999038	.9999033	.9999018	.9998989	.9998945	.9998878	.9998776	.9998608	.9998276	.9997426	.9995020
4.0	.9999367	.9999363	.9999353	.9999334	.9999305	.9999261	.9999195	.9999085	.9998870	.9998309	.9996645
4.1	.9999587	.9999585	.9999578	.9999566	.9999547	.9999519	.9999475	.9999404	.9999266	.9998900	.9997763
4.2	.9999733	.9999732	.9999727	.9999720	.9999707	.9999689	.9999661	.9999616	.9999527	.9999292	.9998523
4.3	.9999829	.9999828	.9999826	.9999821	.9999813	.9999801	.9999783	.9999754	.9999698	.9999548	.9999034
4.4	.9999892	.9999891	.9999889	.9999886	.9999881	.9999874	.9999863	.9999845	.9999809	.9999715	.9999375
4.5	.9999932	.9999932	.9999931	.9999929	.9999925	.9999921	.9999914	.9999902	.9999881	.9999822	.9999599
4.6	.9999958	.9999957	.9999957	.9999955	.9999954	.9999951	.9999947	.9999939	.9999926	.9999889	.9999746
4.7	.9999974	.9999974	.9999973	.9999973	.9999971	.9999970	.9999967	.9999963	.9999955	.9999932	.9999840
4.8	.9999984	.9999984	.9999984	.9999983	.9999983	.9999982	.9999980	.9999977	.9999972	.9999959	.9999901
4.9	.9999990	.9999990	.9999990	.9999990	.9999990	.9999989	.9999988	.9999986	.9999983	.9999975	.9999939
5.0	.9999994	.9999994	.9999994	.9999994	.9999994	.9999993	.9999993	.9999992	.9999990	.9999985	.9999963
5.1	.9999997	.9999997	.9999997	.9999996	.9999996	.9999996	.9999996	.9999995	.9999994	.9999991	.9999978
5.2	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999997	.9999997	.9999995	.9999987
5.3	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999998	.9999997	.9999992
5.4	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999995
5.5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	.9999999	.9999999	.9999999	.9999997
5.6								1.0000000	1.0000000	.9999999	.9999998
5.7										1.0000000	.9999999
5.8											1.0000000
5.9											
6.0											

Table 305a. Circular error probability. Argument c is the ratio of the smaller standard deviation to the larger standard deviation. For the argument c and K, the table provides the probability that a point lies within a circle whose center is at the origin and whose radius is K times the larger standard deviation.

P \ c	0. 0	0. 1	0. 2	0. 3	0. 4	0. 5	0. 6	0. 7	0. 8	0. 9	1. 0
.5000	0. 67449	0. 68199	0. 70585	0. 74993	0. 80785	0. 87042	0. 93365	0. 99621	1. 05769	1. 11807	1. 17741
.7500	1. 15035	1. 15473	1. 16825	1. 19246	1. 23100	1. 28534	1. 35143	1. 42471	1. 50231	1. 58271	1. 66511
.9000	1. 64485	1. 64791	1. 65731	1. 67383	1. 69918	1. 73708	1. 79152	1. 86253	1. 94761	2. 04236	2. 14597
.9500	1. 95996	1. 96253	1. 97041	1. 98420	2. 00514	2. 03586	2. 08130	2. 14598	2. 23029	2. 33180	2. 44775
.9750	2. 24140	2. 24365	2. 25053	2. 26255	2. 28073	2. 30707	2. 34581	2. 40356	2. 48494	2. 58999	2. 71620
.9900	2. 57583	2. 57778	2. 58377	2. 59421	2. 60995	2. 63257	2. 66533	2. 71515	2. 79069	2. 89743	3. 03485
.9950	2. 80703	2. 80883	2. 81432	2. 83289	2. 83830	2. 85894	2. 88859	2. 93347	3. 00431	3. 11073	3. 25525
.9975	3. 02334	2. 02500	3. 03010	3. 03898	3. 05234	3. 07144	3. 09871	3. 13969	3. 20586	3. 31099	3. 46164
.9990	3. 29053	3. 29206	3. 29673	3. 30489	3. 31715	3. 33464	3. 35949	3. 39647	3. 45698	3. 55939	3. 71692

Table 305b. Factors for conversion of probability ellipse to circle of equivalent probability.

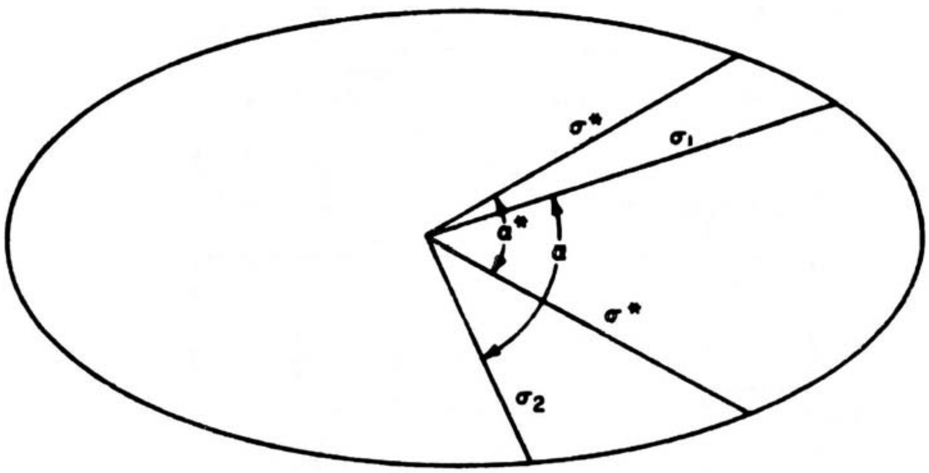

Figure 305f. Transformed parameters of error ellipse.

Utilizing these simplified equations, significant parameters of error ellipses are tabulated in Table 306a as a function of the angle of cut α. Using the CEP curve of Figure 305g, values of the CEP are calculated for each angle, showing that the CEP increases as the angle of cut decreases. The last column in the table gives the factor by which the CEP for angles less than 90° is greater than the CEP for a right angle. This magnification of error curve is plotted in Figure 306b. The curve for the 90 percent probability circle has a slightly differing shape from the CEP curve as shown in Figure 306b. Values for the 90 percent probability circle are given in table Table 306c. Figure 306b indicates the magnitude of the growth of error as the angle of cut varies from 90°.

It is also of interest to consider what values of probability result if the radius of the circle is held constant at the minimum value corresponding to that obtained for the 90° angle of cut. These values may be obtained from the probability versus angle of cut curves in .

Along the ordinate $R/\sigma = 1.177$ which corresponds to the CEP for the circular case, one may read the lesser values of probability corresponding to the various angles of cut. Likewise, one may also obtain the probability values corresponding to holding a circle the size of the 90 percent probability circle for the circular case by using the ordinate $R/\sigma = 2.15$ (also equivalent to 1.82 times the CEP). These two curves are plotted in Figure 306e and the numerical values are given in Table 306d. It is to be noted that the probability values are not inversely related to the error factors plotted in the preceding curves. The geometric error factor is a simple trigonometric function; the probability curves are exponential functions.

307. Clarification of Terminology

The following discussion is presented to insure that there is no misunderstanding with respect to the use of terms having one meaning when discussing one-dimensional errors and another when discussing two-dimensional errors.

Although the basic problem of position location is concerned with the two dimensions necessary to describe an area, one-dimensional error measures are commonly applied to each of the two dimensions involved. As demonstrated in article 305, the use of the one-dimensional standard deviation of each line of position permitted a general approach to the consideration of the error ellipse.

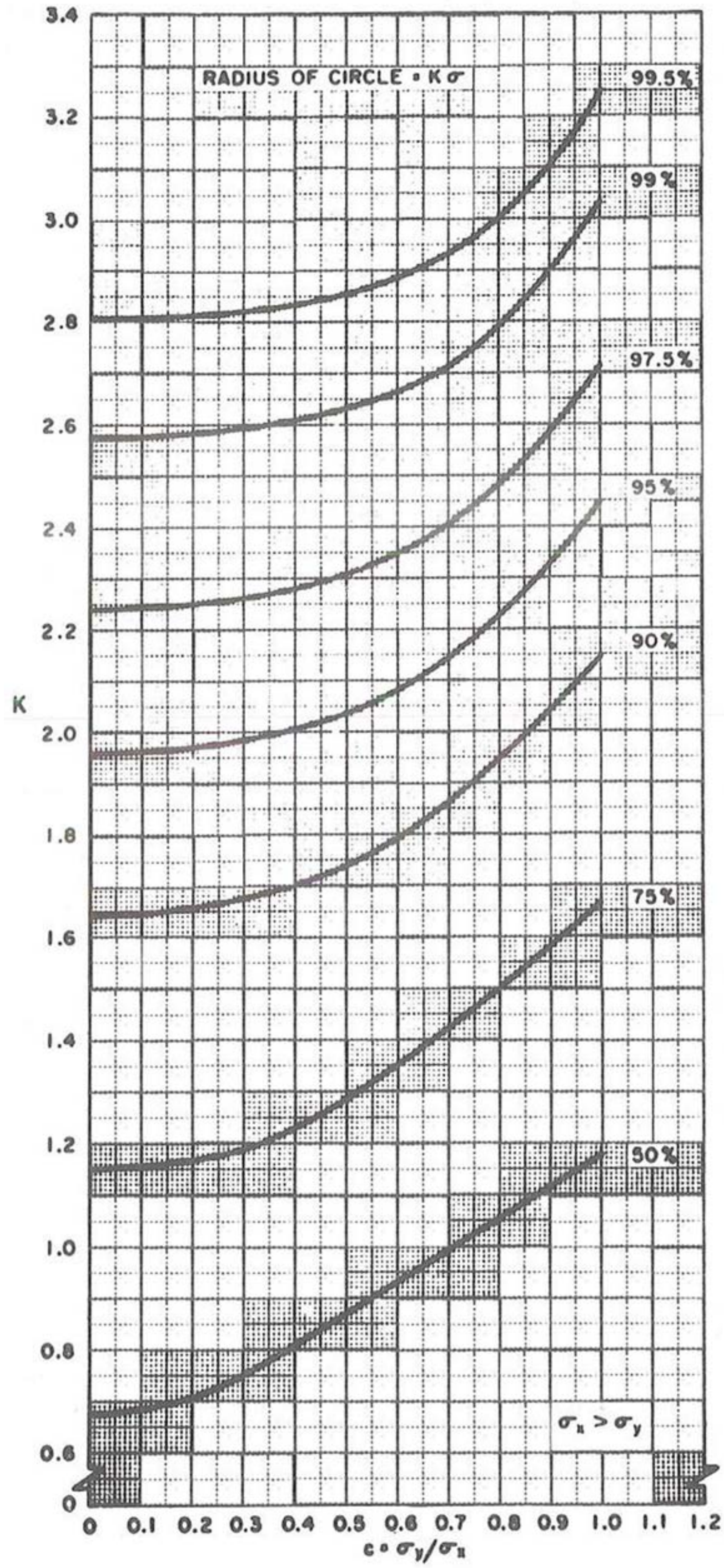

Figure 305g. Factors for conversion of probability ellipse to circle of equivalent probability.

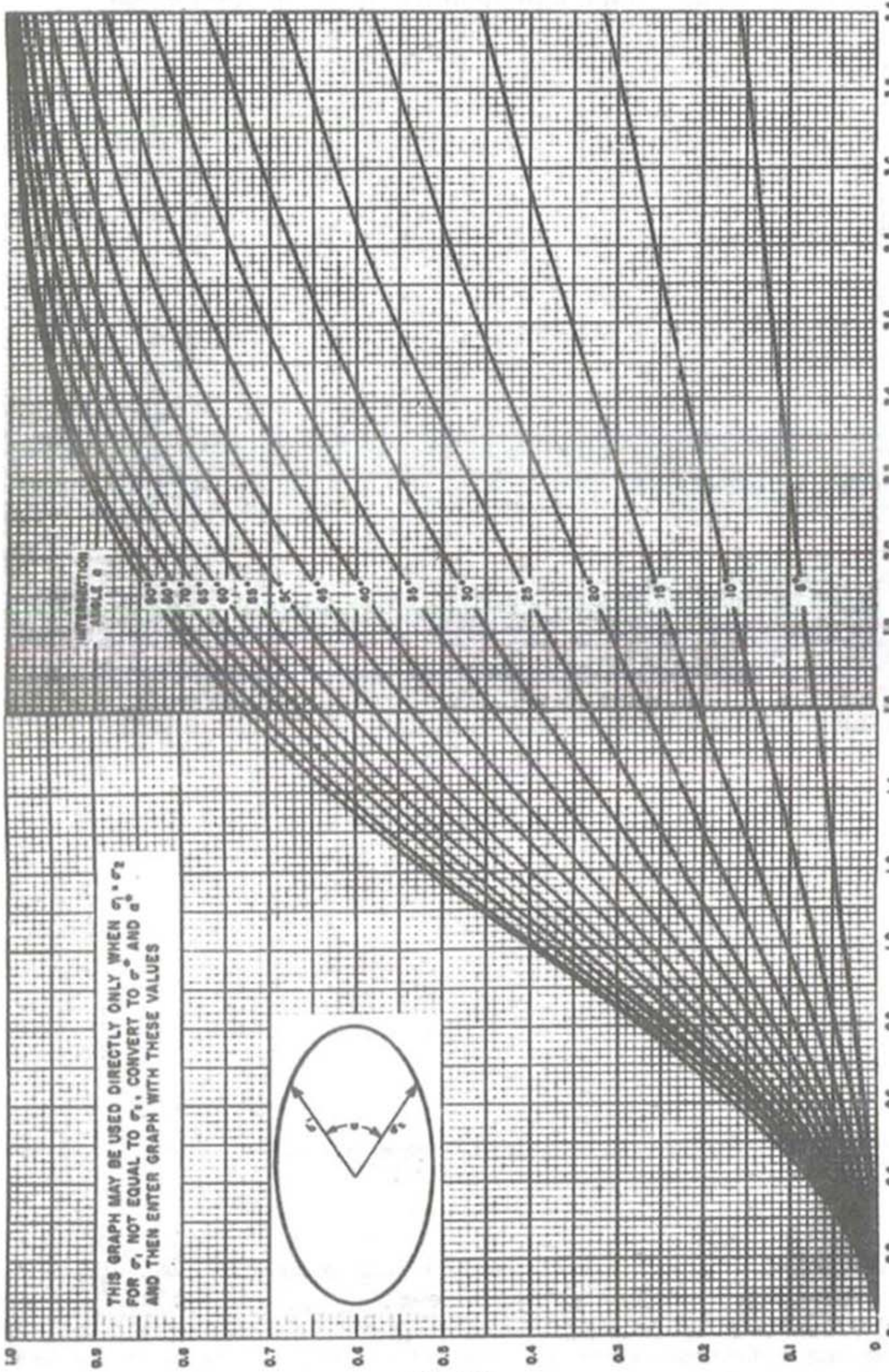

Figure 305h. Probability versus the radius of the circle divided by the standard error and the angle of cut for elliptical bivariate distributions with two equal standards deviations.

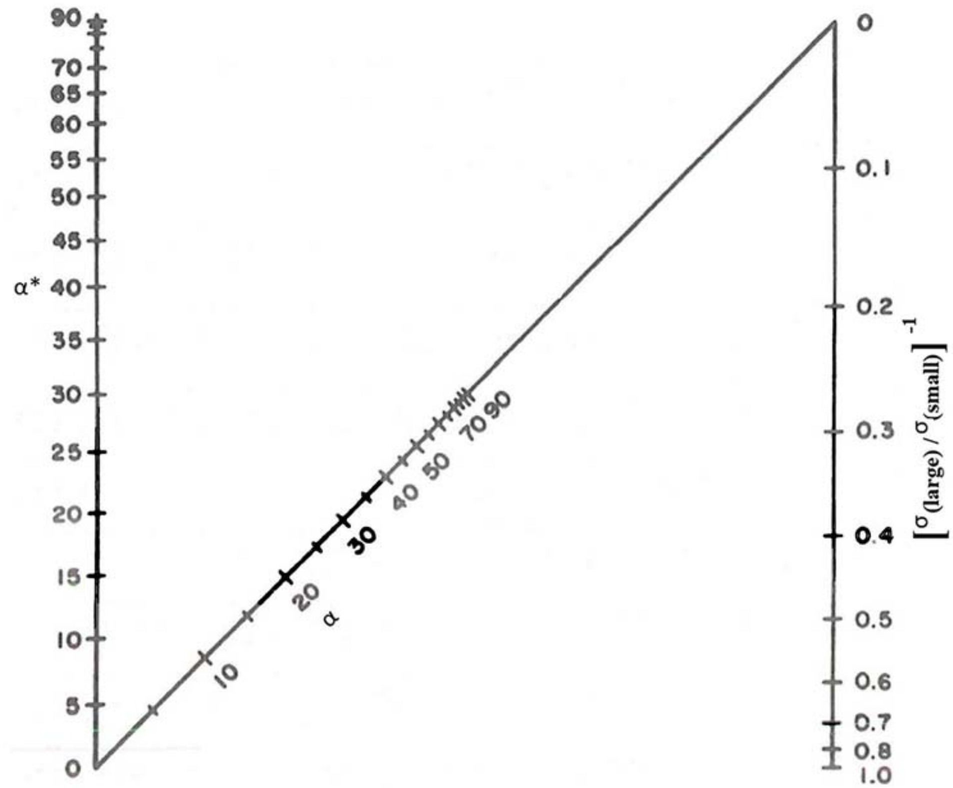

*Figure 305i. Nomogram to obtain α *.*

308. One-Dimensional Errors

The terms **standard deviation**, **sigma** (σ), and **root mean square (RMS) error** have the same meaning in reference to one-dimensional errors. The basic equation of the normal (Gaussian) distribution indicates the use of the Greek letter sigma, σ, from which its use for standard deviation arises:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty$$

where the Greek letter μ is the mean of the distribution.

Standard deviation of a measurement system is a property that may be determined experimentally. If a large number of measurements of the same quantity, a length for example, are made and compared with their mean value, the standard deviation is the square root of the sum of the squares of the differences (deviations) of the measurements from the mean value divided by one less than the number of measurements taken. The mean, or average value, is the sum of the measurements divided by the number of the measurements. Symbolically this operation is represented as:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu)^2}{n-1}}, \quad \mu = \frac{\sum_{i=1}^{n} x_i}{n}$$

The term root-mean-square (RMS) error comes from this latter method of computation.

Numerically, the values between the mean plus or minus one sigma (one standard deviation) corresponds to 68.27 percent of the distribution. That is, if a large number of measurements were made of a given quantity, 68.27 percent of the errors would be within the value of the mean plus or minus one standard deviation, or within $\mu \pm 1\sigma$. Likewise, errors within $\mu \pm 2\sigma$ correspond to 95.45 percent of the total errors and errors within $\mu \pm 3\sigma$ correspond to 99.73 percent of the total errors. Colloquially, these conditions are described as not exceeding the one-, two-, and three-sigma values, respectively.

The term probable error is identical in concept to standard deviation. The term differs from standard deviation in that it refers to the median error; that is, no more than half the errors in the measurement sample are greater than the value of the probable error. Linear probable error is related to standard deviation by a multiplication factor (Table 308a). One probable error equals 0.6745 times one standard deviation.

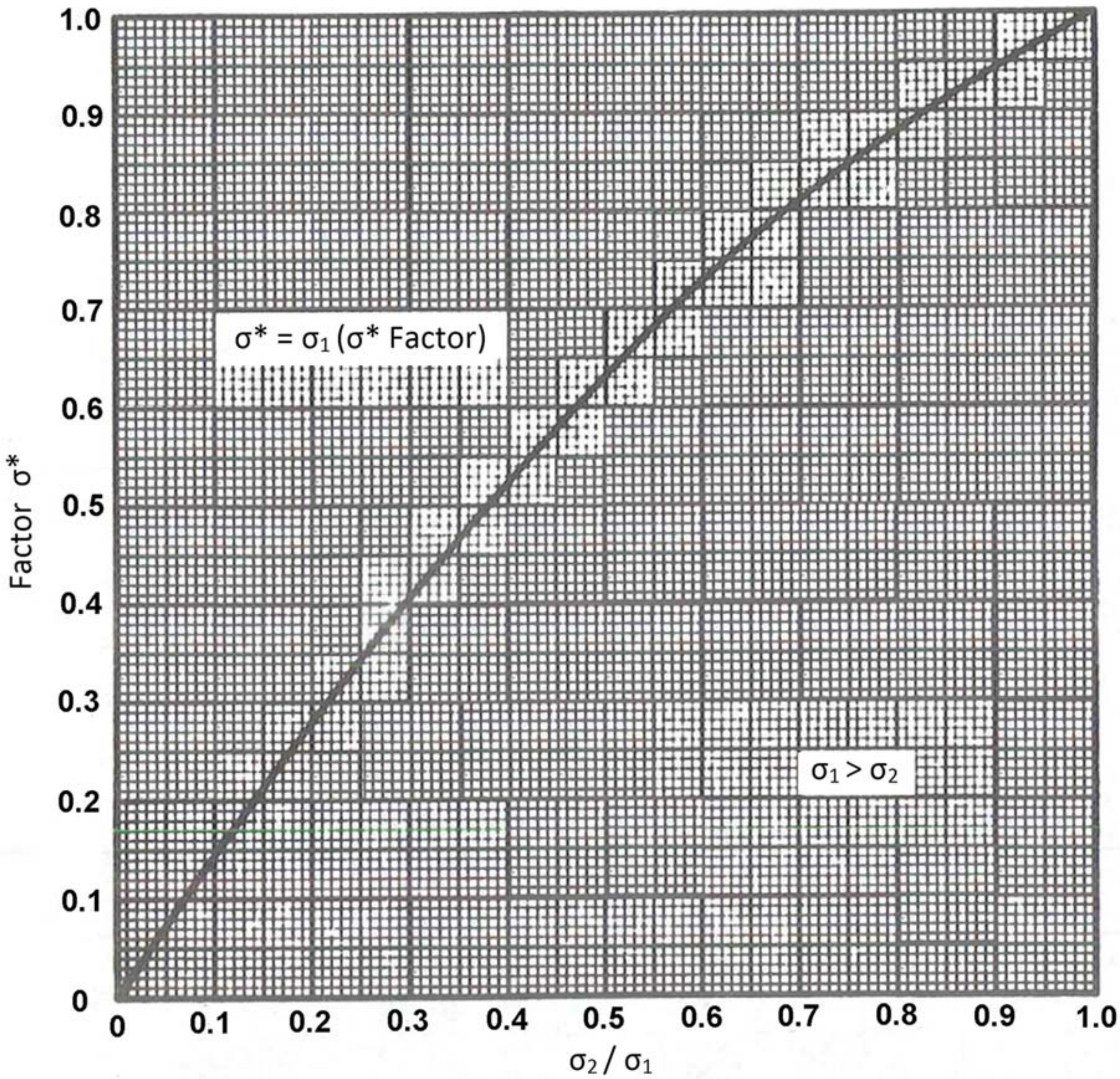

Figure 305j. σ ** factors versus /* σ_2/σ_1 *ratio.*

α	σ_x	σ_y	c	K	CEP	Error Factor
90	1.0	1.0	1.0	1.177	1.177	1.00
80	1.10	0.924	0.839	1.078	1.186	1.01
70	1.234	0.865	0.700	0.996	1.228	1.042
60	1.414	0.817	0.577	0.914	1.292	1.099
50	1.672	0.782	0.466	0.847	1.420	1.206
45	1.847	0.766	0.414	0.815	1.508	1.281
40	2.06	0.753	0364	0.783	1.620	1.376
30	2.74	0.733	0.268	0.734	2.01	1.710
20	4.06	0.718	0.176	0.700	2.85	2.42
10	8.11	0.710	0.087	0.680	5.52	4.69

Table 306a. Significant parameters of error ellipses when $\sigma_1 = \sigma_2$

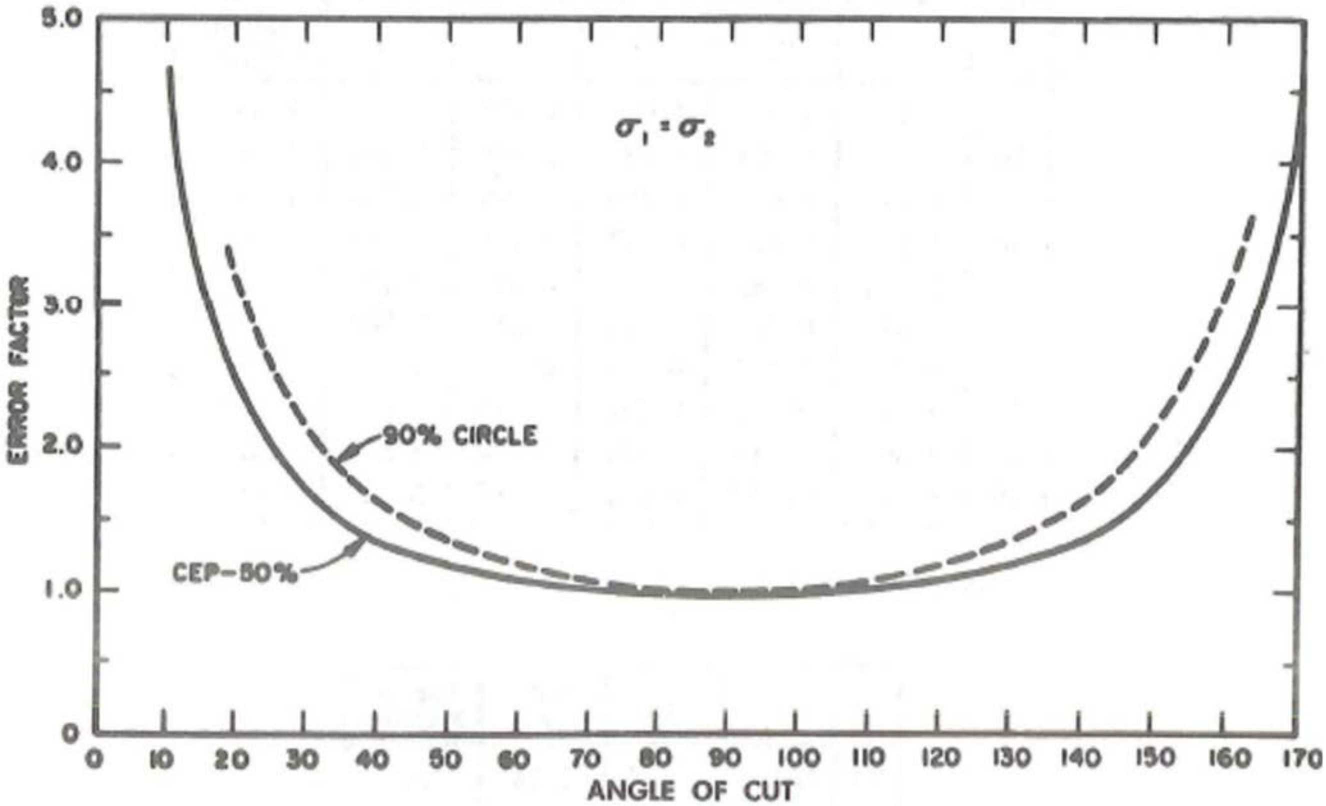

Figure 306b. CEP magnification versus angle of cut.

α	*c*	*K*	90% R	Error Factor
90	1.0	2.145	2.145	1.00
80	0.839	1.98	2.18	1.015
70	0.700	1.86	2.30	1.07
60	0.577	1.775	2.51	1.7
50	0.466	1.72	2.88	1.34
45	0.414	1.702	3.15	1.47
40	0.364	1.687	3.47	1.615
30	0.268	1.665	4.53	2.11
20	0.176	1.652	6.72	3.13
10	0.087	1.645	13.35	6.22

Table 306c. 90 percent error factor

α	*P*	*P*
90	50	90
80	49.4	89.2
70	47.5	86.9
60	44.0	82.4
50	39.5	76
40	37	66
30	25	53
20	17	37
10	8	19

Table 306d. Probability decrease with decreasing angle of cut for a circle of constant radius

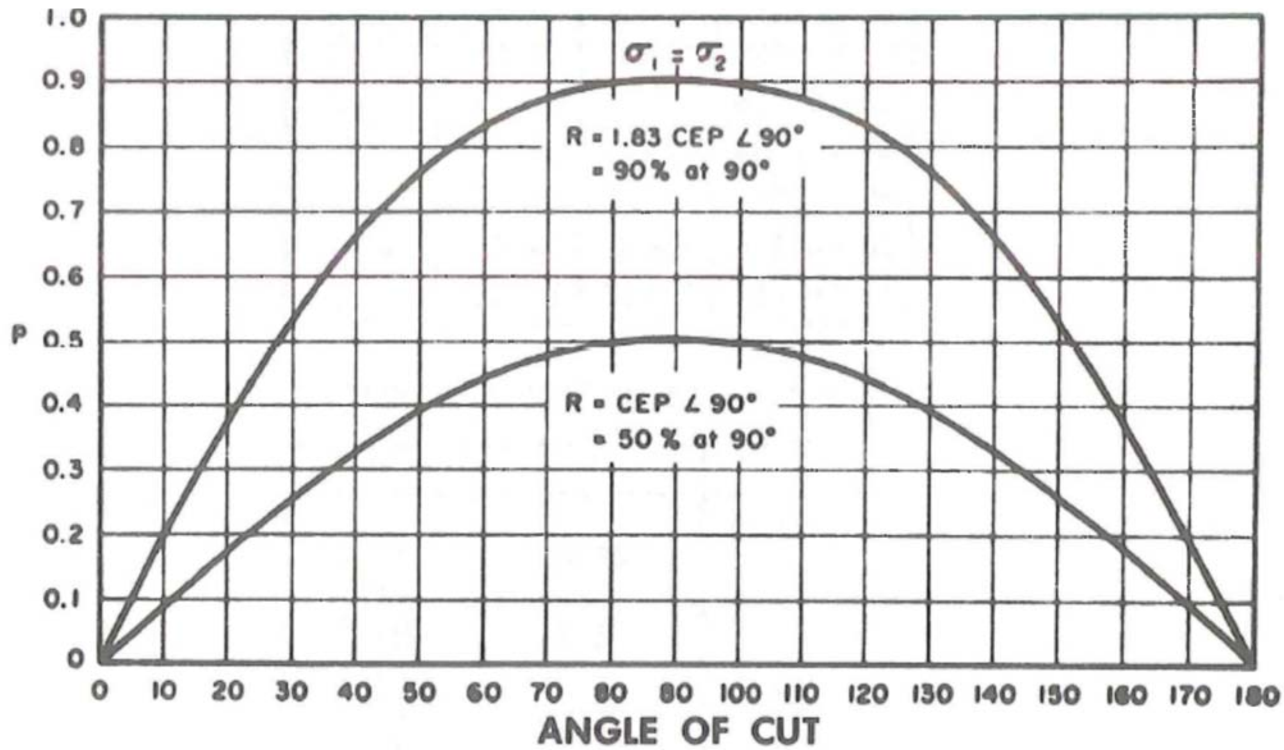

Figure 306e. Decrease in probability for a circle of constant radius versus angle of cut.

The term **variance** is met most frequently in detailed mathematical discussions.

From/To	**50.00%**	**68.27%**	**95.00%**	**99.73%**
50.00%	1.0000	1.4826	2.9059	4.4475
68.27%	0.6745	1.0000	1.9600	3.0000
95.00%	0.3441	0.5102	1.0000	1.5307
99.73%	0.2248	0.3333	0.6533	1.0000

Table 308a. Linear error conversion factors.

309. Two-Dimensional Error

Terms similar or identical in words to those used for one-dimensional error descriptions are also used with two-dimensional or **bivariate error** descriptions. However, in the two-dimensional case, not all of these terms have the same meaning as before; considerable care is needed to avoid confusion.

Standard deviation or **sigma** has a definable meaning only in the specific case of the circular normal distribution where $\sigma_x = \sigma_y$:

$$P_R = 1 - e\frac{R^2}{2\sigma^2}$$

In the case of the circular normal distribution, the standard deviation σ is equivalent to the standard deviation along both orthogonal axes. Because of concern with a radial distribution, the total distribution of errors involves numbers different from those of the linear case (Table 308a and Table 309a). In the circular case, 1σ error indicates that 39.35 percent of the errors would not exceed the value of the 1σ error; 86.47 percent would not exceed the 2σ error; 98.89 percent would not exceed the 3σ error; and 99.78 percent would not exceed the 3.5σ error.

From/To	**39.35%**	**50.00%**	**63.21%**	**95.00%**	**99.78%**
39.35%	1.0000	1.1774	1.4142	2.4477	3.5000
50.00%	0.8493	1.0000	1.2011	2.0789	2.9726
63.21%	0.7071	0.8325	1.0000	1.7308	2.4749
95.00%	0.4085	0.4810	0.5778	1.0000	1.4299

Table 309a. Circular error conversion factors.

From/To	**39.35%**	**50.00%**	**63.21%**	**95.00%**	**99.78%**
99.78%	0.2857	0.3364	0.4040	0.6993	1.0000

Table 309a. Circular error conversion factors.

Because the usual case where there are two-dimensional distributions is that the standard deviations are different, resulting in an elliptical distribution, the circular standard deviation is less useful than the linear standard deviation. It is more common to describe two-dimensional distributions by the two separate one-dimensional standard deviations associated with each error axis. References, however, often do not make this distinction, referring to the position accuracy of a system as 600 feet (2σ), for example. Such a description should leave the reader wondering whether the measure is circular error, in which case the numbers describe the 86 percent probability circle, or whether the number are to be interpreted as one-dimensional sigmas along each axis, in which case the 95 percent probability circle is indicated (assuming the distribution to be circular, which actually it may not be).

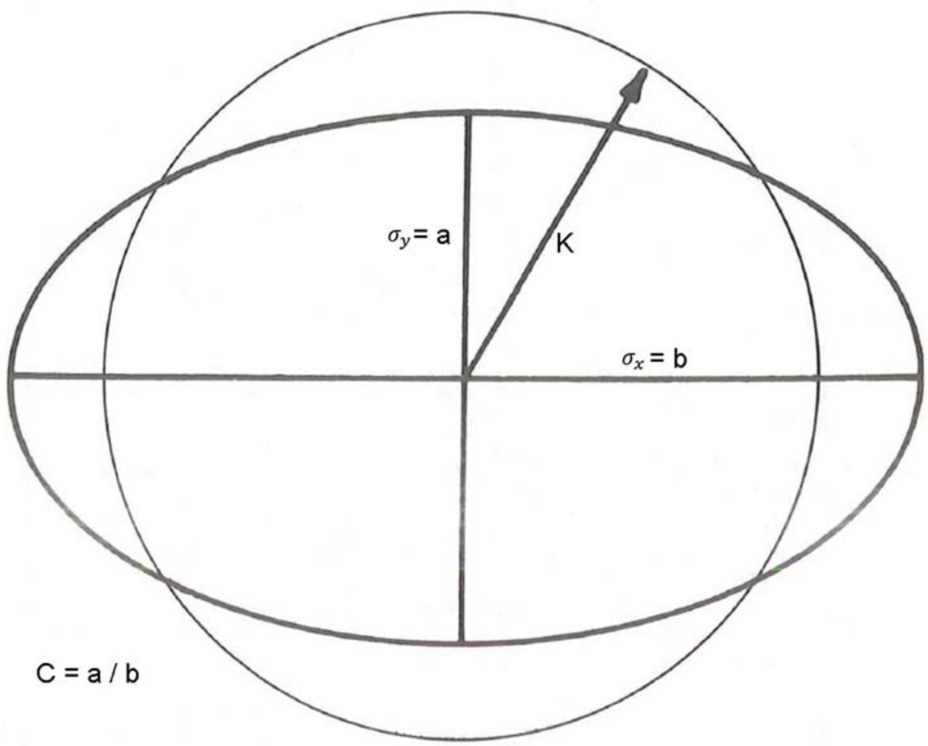

Figure 309b. Error ellipse and circle of equivalent probability.

The term **RMS (root mean square) error** when applied to two-dimensional errors does not have the same meaning as standard deviation. The term has the same meaning as radial error or d_{rms}, discussed later. Such use of the term is deprecated.

In a circular normal distribution, the term **circular probable error (CPE)** or **circular error probable (CEP)** refers to the radius of the circle inside of which there is a 50 percent probability of being located.

The term CEP is also used to indicate the radius of a circle inside of which there is a 50 percent probability of being located, even though the actual error figure (Figure 309b) is an ellipse. Article 305 describes one of the methods of obtaining such CEP equivalents when given ellipses of varying eccentricities. Curves and tables are available for performing this calculation. Despite the availability of these curves and tables, approximations are often made for this calculation of a CEP when the actual error distribution is elliptical. Several of these approximations are indicated and plotted for comparison with the exact curve in Figure 309c. Of the various approximations shown, the top curve, the one which diverges the most rapidly, appears to be the most commonly used.

Another factor of interest concerning the relationship of the CEP to various ellipses is that the area of the CEP circle is always greater than the basic ellipse. Table 309d indicates that the divergence between the actual area of the ellipse of interest and the circle of equivalent probability increases as the ellipse becomes thinner and more elongated.

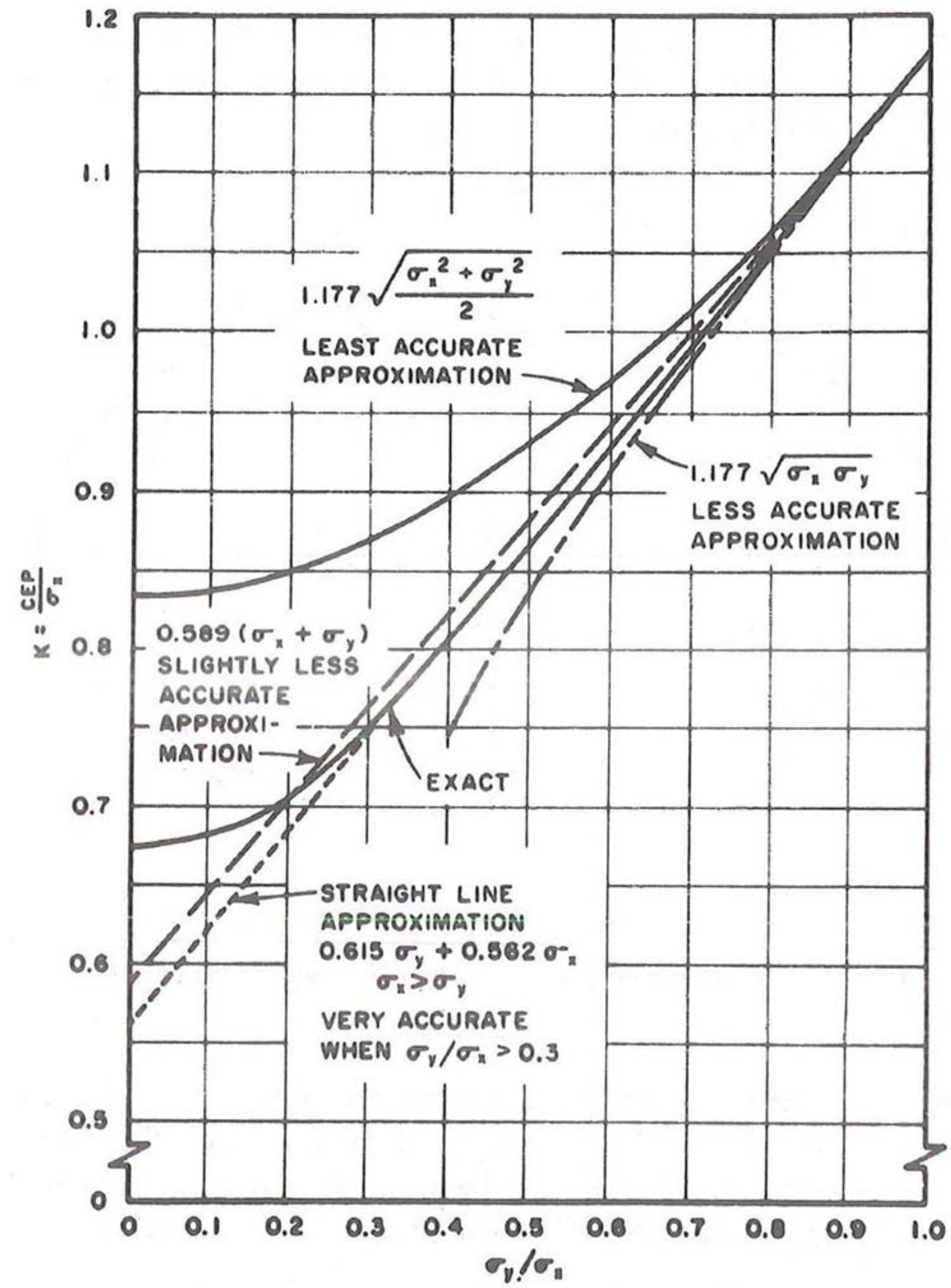

Figure 309c. CEP for elliptical error distribution approximations.

C = a / b	Area of 50% ellipse	Area of equivalent circle
0.0	0	1.43
0.1	0.437	1.46
0.2	0.874	1.56
0.3	1.31	1.76
0.4	1.75	2.06

Table 309d. Comparison of areas of 50% ellipses of varying eccentricities with areas of circles of equivalent probabilities.

C = a / b	Area of 50% ellipse	Area of equivalent circle
0.5	2.08	2.37
0.6	2.62	2.74
0.7	3.06	3.12
0.8	3.49	3.52
0.9	3.93	3.94
1.0	4.37	4.37

Table 309d. Comparison of areas of 50% ellipses of varying eccentricities with areas of circles of equivalent probabilities.

The value of the CEP may be related to the radius of other values of probability circles analytically for the case of the circular normal distribution by solving the basic equation for various values of probability. For this special case of the circular normal distribution, these relationships are shown drawn to scale in Figure 309e with the associated values tabulated in Table 309f.

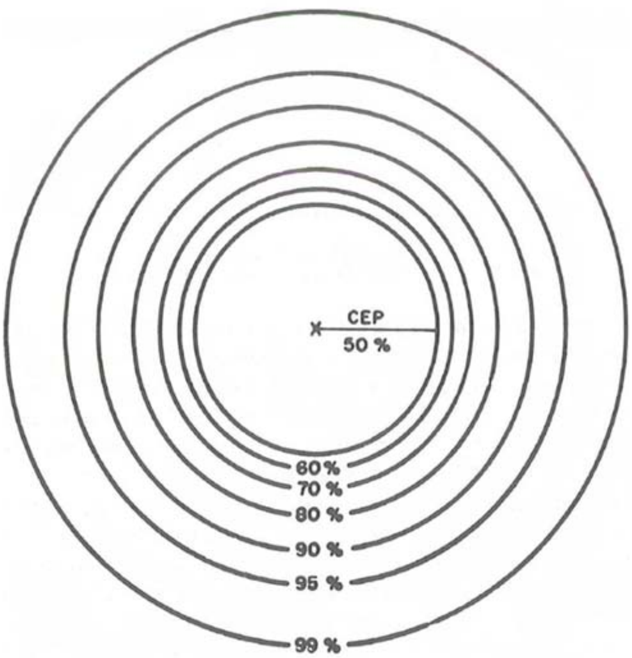

Figure 309e. Relationship between CEP and other probability circles.

Multiply values of CEP by	To obtain radii of circle of probability
1.150	60%
1.318	70%

Table 309f. Relationship between CEP and radii of other probabilities circles of the circular normal distribution.

Multiply values of CEP by	To obtain radii of circle of probability
1.414	75%
1.524	80%
1.655	85%
1.823	90%
2.079	95%
2.578	99%

Table 309f. Relationship between CEP and radii of other probabilities circles of the circular normal distribution.

The derivation of these values is shown in the following analysis. First, the factor relating the CEP to the circular sigma is derived, then, as a second example, the relationship between the 75 percent probability circle and the circular sigma is derived. The ratio of these two values is then the value shown in Table 309f for the 75 percent value.

The circular normal distribution equation is:

$$P_R = 1 - e - \frac{R^2}{2\sigma^2} ,$$

and

$$CEP = P(R) = 0.5$$

$$1 - e - \frac{R^2}{2\sigma^2} = 0.5$$

$$e - \frac{R^2}{2\sigma^2} = 0.5 .$$

Take the natural logarithm of both sides

$$\ln\left(e - \frac{R^2}{2\sigma^2}\right) = \ln 0.5$$

$$\frac{R^2}{2\sigma^2} = \ln 2 \qquad (\ln 0.5 = -\ln 2)$$

$$R = 1.1774\sigma .$$

For the 75 percent probability circle,

$$1 - e{-}\frac{R^2}{2\sigma^2} = 0.75$$

$$e{-}\frac{R^2}{2\sigma^2} = 0.25$$

$$\ln\left(e{-}\frac{R^2}{2\sigma^2}\right) = \ln 0.25$$

$$\frac{R^2}{2\sigma^2} = \ln 4$$

$$R = 1.665\sigma$$

$$\frac{R(75\%)}{R(50\%)} = \frac{1.665\sigma}{1.177\sigma} = 1.414 .$$

The factors tabulated in Table 309f are sometimes used to relate varying probability circles when the basic distribution is not circular, but elliptical. That such a procedure is inaccurate may be seen by the curves of . It can be seen that the errors involved are small when the eccentricities are small. But the errors increase significantly when both high values of probability are desired and when the ellipticity increases in the direction of long, narrow distributions.

The terms **radial error, root mean square error,** and $\boldsymbol{d_{rms}}$ are identical in meaning when applied to two-dimensional errors. Figure 309h illustrates the definition of d_{rms}. It is seen to be the square root of the sum of the square of the 1 sigma error components along the major and minor axes of a probability ellipse. The figure details the definition of 1 d_{rms}. Similarly, other values of d_{rms} can be derived by using the corresponding values of sigma. The measure d_{rms} is not equal to the square root of the sum of the squares of σ_1 and σ_2 that are the basic errors associated with the lines of position of a particular measuring system. The procedures described in section 305 must first be utilized to obtain the values shown as σ_x and σ_y.

The three terms (radial error, root-mean-square error, and d_{rms}) used as a measure of error are somewhat confusing because they do not correspond to a fixed value of probability for a given value of the error measure. The terms can be conveniently related to other error measures only when $\sigma_x = \sigma_y$, and the probability figure is a circle. In the more common elliptical cases, the probability associated with a fixed value of d_{rms} varies as a function of the eccentricity of the ellipse. One d_{rms} is defined as the radius of the circle obtained when $\sigma_x = 1$, in Figure 309h, and σ_y varies from 0 to 1. Likewise, 2 d_{rms} is the radius of the circle obtained when $\sigma_x = 2$, and σ_y varies from 0 to 2. Values of the length of the radius d_{rms} can be calculated as shown in Table 309j. From these values the associated

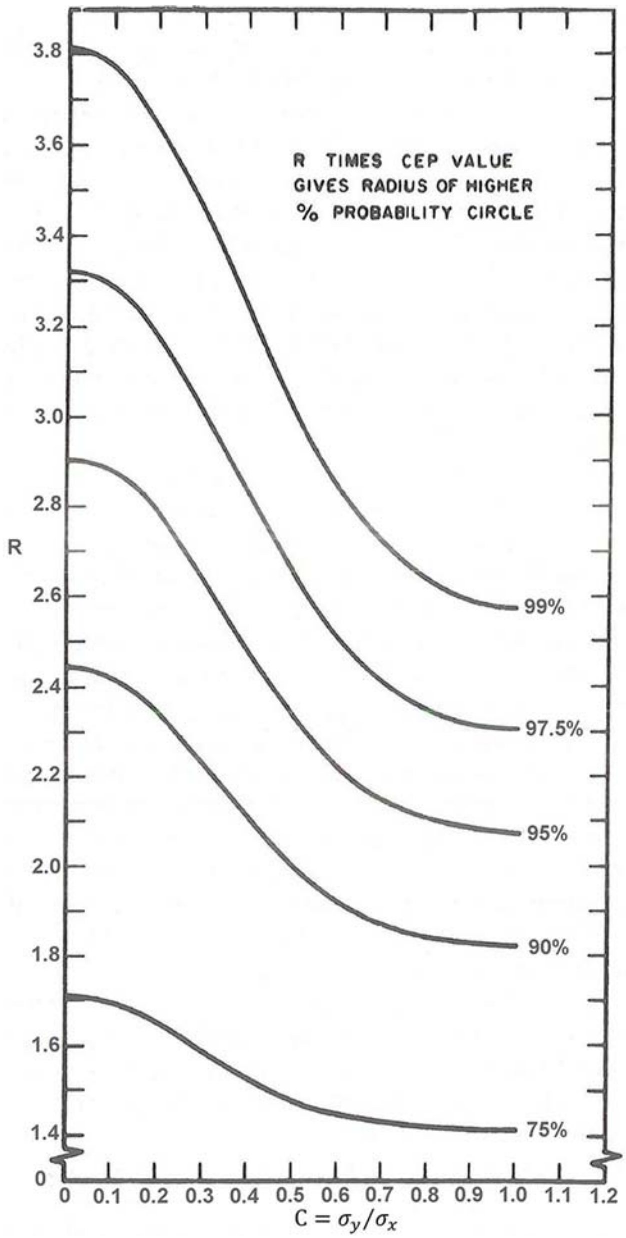

Figure 309g. Relation of probability circles to CEP versus ellipticity.

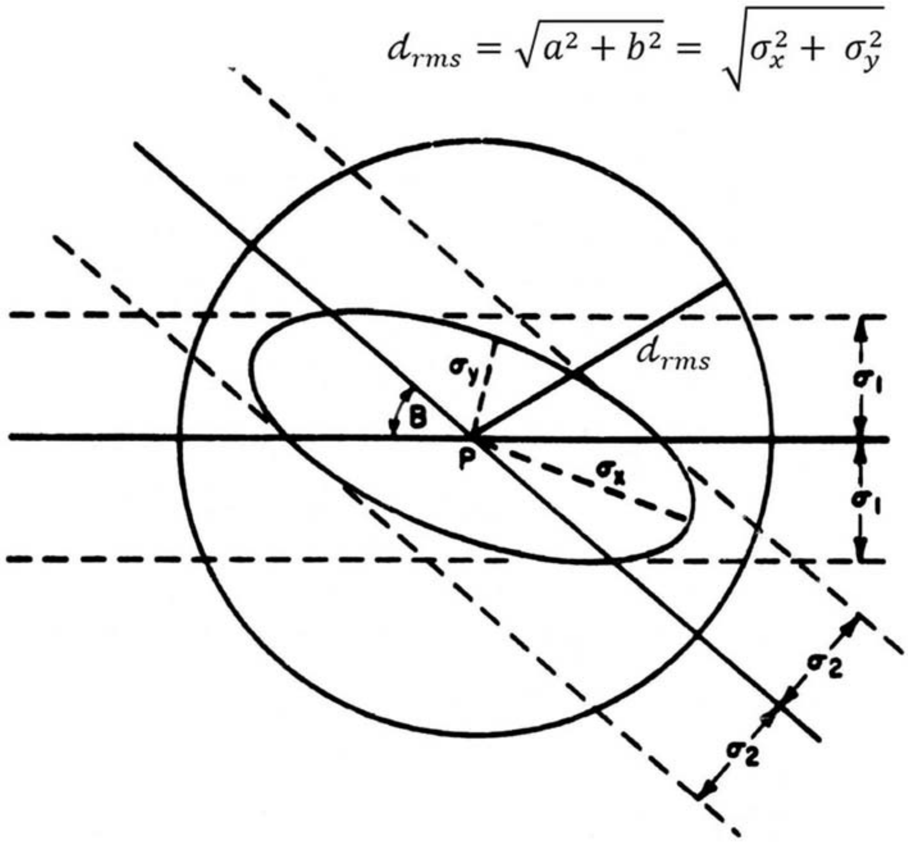

Figure 309h. CEP for elliptical error distribution approximations.

probabilities can be determined from the tables of section 305. The variations of probability associated with the values of 1 d_{rms} and 2 d_{rms} are shown in the curves of and . shows the lack of a constant relationship in a slightly different way. Here the ratio d_{rms}/CEP is plotted against the same measure of ellipticity. The three figures show graphically that there is not a constant value of probability associated with a single value of d_{rms}.

Figure 309i shows the substitution of the circular form for elliptical error distributions. When σ_x and σ_y are equal, the probability represented by 1 d_{rms} is 63.21 percent. When σ_x and σ_y are unequal (σ_x being the greater value), the probability varies from 64 percent when $\sigma_y/\sigma_x = 0.8$ to 68 percent when $\sigma_y/\sigma_x = 0.3$.

310. Navigation System Accuracy

In a navigation system, **predictability** is the measure of the accuracy with which the system can define the position in terms of geographical coordinates; **repeatability** is the measure of the accuracy with which the system permits the user to return to a position as defined only in terms of the coordinates peculiar to that system. **Predictable accuracy**, therefore, is the accuracy of positioning with respect to geographical coordinates; **repeatable accuracy** is the accuracy with which the user can return to a position whose coordinates have been measured previously with the same system. For example, the distance specified for the repeatable accuracy of a system such as GPS is the distance between two GPS positions established using the same satellites at different times. The correlation between the geographical coordinates and the system coordinates may or may not be known.

Relative accuracy is the accuracy with which a user can determine their position relative to that of another user of the same navigation system at the same time. Hence, a system with high relative accuracy provides good rendezvous capability for the users of the system. The correlation between the geographical coordinates and the system coordinates is not relevant.

311. Most Probable Position

Some navigators, particularly those of little experience, have been led by the simplified definitions and explanations usually given in texts to conclude that the line of position is infallible, and that a fix is without error, overlooking the frequent incompatibility of these two notions. Too often the idea has prevailed that information is either all right or all

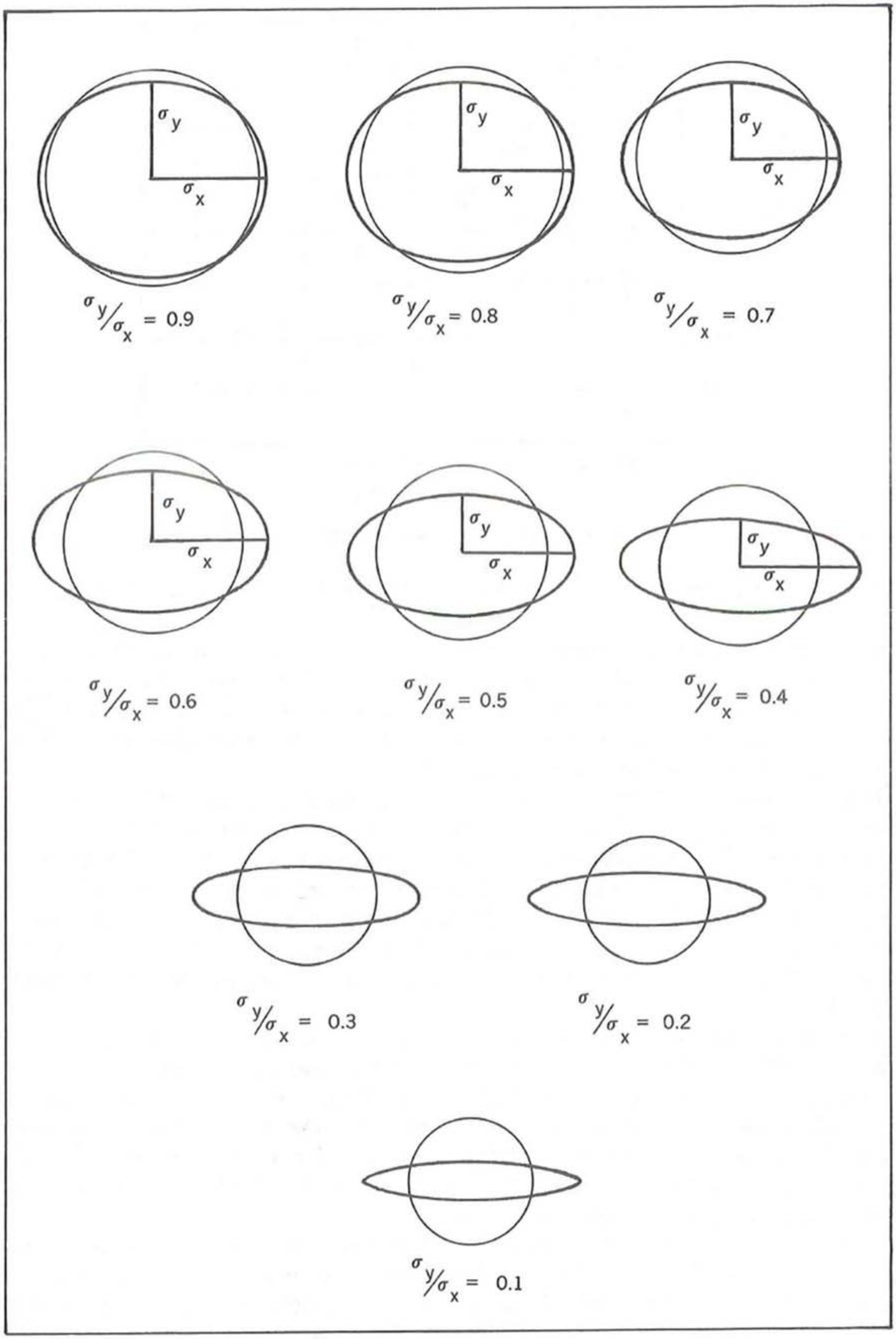

Figure 309i. Substitution of the circular form for elliptical error distributions.

σ_y	σ_x	LENGTH OF 1 d_{rms}	PROBABILITY	
			1 $_{drms}$	2 d_{rms}
0.0	1.0	1.000	0.683	0.954
0.1	1.0	1.005	0.682	0.955
0.2	1.0	1.020	0.682	0.957
0.3	1.0	1.042	0.676	0.961
0.4	1.0	1.077	0.671	0.966
0.5	1.0	1.118	0.662	0.969
0.6	1.0	1.166	0.650	0.973
0.7	1.0	1.220	0.641	0.977
0.8	1.0	1.280	0.635	0.980
0.9	1.0	1.345	0.632	0.981
1.0	1.0	1.414	0.632	0.982
$d_{rms} = \sqrt{\sigma_x^2 + \sigma_y^2}$ when σ_x and σ_y are at right angles to each other.				

Table 309j. Calculations of d_{rms}.

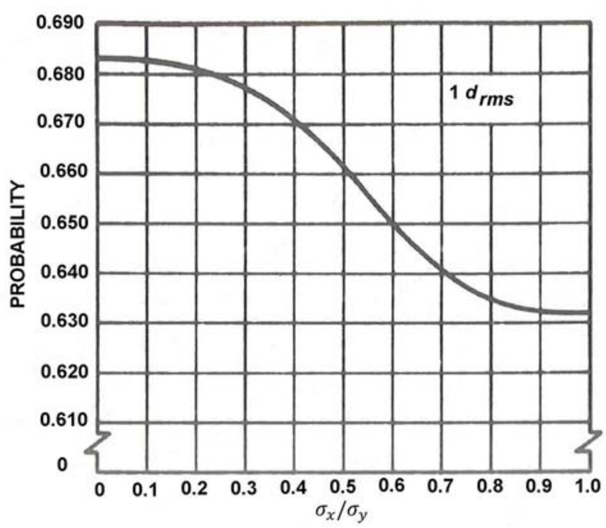

Figure 309k. Variation in d_{rms} with ellipticity (1 d_{rms})..

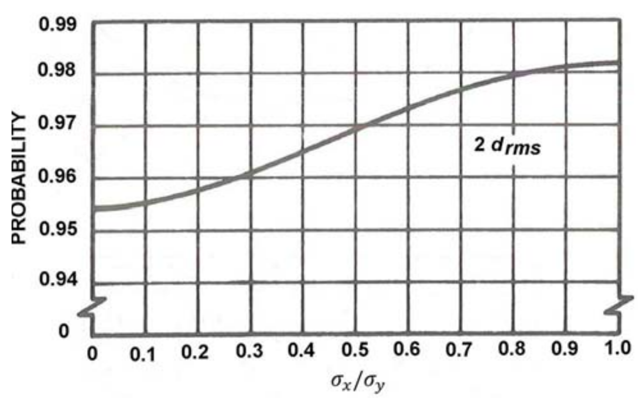

Figure 309l. Variation in d_{rms} with ellipticity (2 d_{rms}).

wrong. An example is the practice of establishing an estimated position at the foot of the perpendicular from a dead reckoning position to a line of position. The assumption is that the vessel *must* be somewhere on the line of position. The limitations of this often valuable practice are not understood by these inexperienced navigators.

A more realistic concept is that of the **most probable position (MPP)**, which recognizes the probability of error in *all* navigational information, and determines position by an evaluation of all available information, using the principles of errors.

Suppose a vessel were to start from a completely accurate position and proceed on dead reckoning. If course and speed over the bottom were of equal accuracy, the uncertainty of dead reckoning positions would increase equally in all directions with either distance or elapsed time (for any one speed these would be directly proportional and therefore either could be used). Therefore, a circle of uncertainty would grow around the dead reckon-

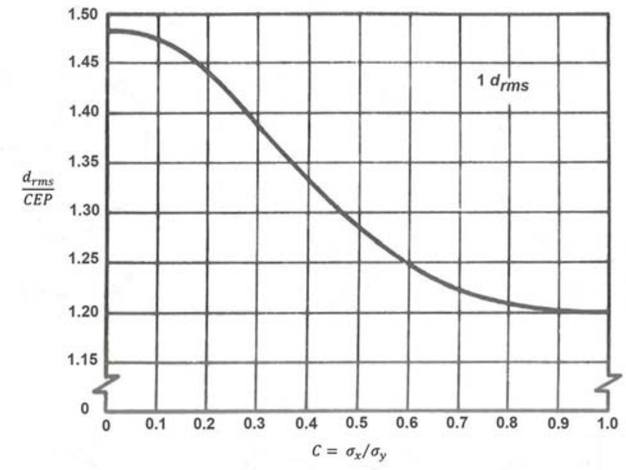

Figure 309m. Ellipticity versus d_{rms} /CEP (1 d_{rms}).

ing position as the vessel proceeded. If the navigator had full knowledge of the distribution and nature of the errors of course and speed, and the necessary knowledge of statistical analysis, s/he could compute the radius of the circle of uncertainty, using the 50 percent, 95 percent, or other probabilities.

In ordinary navigation, this is not practicable, but based upon experience and judgment, the navigator might estimate at any time the likely error of his or her dead reckoning or estimated position. With practice, navigators might acquire considerable skill in making this estimate. They would take into account, too, the fact that the area of uncertainty might be better represented by a circle, the major axis being along the course line if the estimated error of the speed were greater than that of the course, and the minor axis being along the course line if the estimated error of the course were greater. They would recognize, too, that the size of the area of uncertainty would not grow in direct proportion to the distance or elapsed time, because disturbing factors such as wind and current could not be expected to remain of constant magnitude and direction. Also, they would know that the starting point of the dead reckoning would not be completely free from error.

At some future time additional positional information would be obtained. This might be a line of position from a celestial observation. This, too, would be accompanied by an estimated error which might be computed for a certain probability if the necessary information and knowledge were available. If the dead reckoning had started from a good position obtained by means of landmarks, the likely error of the initial position would be very small. At first the dead reckoning or estimated position would probably be more reliable than a line of position obtained by celestial observation. But at *some* distance the two would be equal, and beyond this the line of position might be more accurate.

The determination of most probable position does depend upon *which* information is more accurate. In Figure 311a a dead reckoning position, $\mu_1 = 0.6$, is shown surrounded by a circle of uncertainty with one-sigma error σ_1. A line of position is also shown, with its area of uncertainty with one-sigma error σ_2. The most probable position is within the overlapping area, and if the uncertainty of the dead reckoning position and that of the line of position are about equal, it might be taken at the center of the line perpendicular to the line of position that runs through the dead reckoning position. The intersection of the line of position with the perpendicular is position $\mu_2 = 0.5$. The most probable position means are taken to have only components on the perpendicular. If the overall errors are considered normal, and they are probably approximately, *the effect of each error is proportional to its square, acting on the other position measurement*. Thus, if the likely error of the dead reckoning position is $\sigma_1 = 3$ miles, and that of a line of position is $\sigma_2 =$ 2 miles, the most probable position is nearer the line of position, being given by

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}\mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}\mu_2 =$$

$$\frac{3^2}{3^2 + 2^2}0.5 + \frac{2^2}{3^2 + 2^2}0.6 = \frac{9}{13}0.5 + \frac{4}{13}0.6 \approx 0.53$$

with an uncertainty given by

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

or

$$\sigma = \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2\sigma_2^2}} = \sqrt{\frac{2^2 + 3^2}{2^2 3^2}} = \sqrt{\frac{13}{36}} \approx 0.60$$

showing that the uncertainty of combining the two position estimates results in a position error smaller than that of either of the two contributing errors.

If a fix is obtained from two lines of position, the area of uncertainty is a circle if the lines are perpendicular, have equal likely errors, and these errors can be considered normal. If one is considered more accurate than the other, the area is an ellipse, the two axes being proportional to the standard deviations of the two lines of position. As shown in Figure 311b, it is also an ellipse if the likely error of each is equal and the lines cross at an oblique angle. If the errors are unequal, the major axis of the ellipse is more nearly in line with the line of position having the smaller likely error.

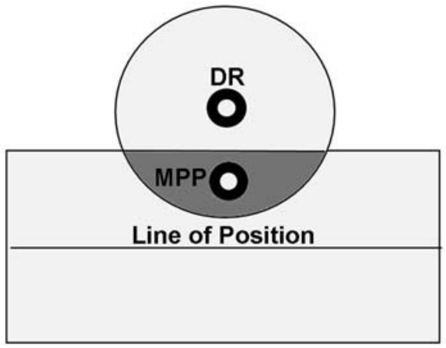

Figure 311a. A most probable position based upon a dead reckoning position and line of position having equal probable errors.

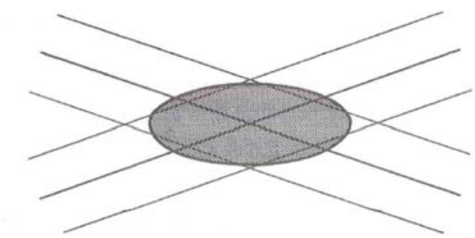

Figure 311b. Ellipse of uncertainty with line of positions of equal probable errors crossing at an oblique angle.

If a fix is obtained from three or more lines of position spread in azimuth by more than 180°, and the error of each line is normal and equal to that of the others, the most probable position is the center of the figure. By "center" is meant that point within the figure which is equidistant from the sides. If the lines are of unequal likely error, the distance of the most probable position from each line of position is proportional to the *square* of the likely error of that line times the sine of the angle formed by the other two lines.

In the discussion of most probable position from lines of position, it has been assumed that no other positional information is available. Usually, this is an incorrect assumption, for there is nearly always a dead reckoning or estimated position. This can be considered in any of several ways. The square of its likely error can be used in the same manner as the square of the likely error of each line of position. A most probable position based upon the dead reckoning or estimated position and the most reliable line of position might be determined as explained above, and that line of position replaced with a new one parallel to it but passing through the most probable position just determined. This adjusted line of position can then be assigned a smaller likely error and used with the other lines of position to determine the overall most probable position. A third way is to establish a likely error for the fix, and consider the most probable position as that point along the straight line joining the fix and the dead reckoning or estimated position, the relative distances being equal to the square of the likely error of each position.

The value of the most probable position determined as suggested above depends upon the degree to which the various errors are in fact normal, and the accuracy with which the likely error of each is established. From a practical standpoint, the second factor is largely a matter of judgment based upon experience. It might seem that interpretation of results and establishment of most probable position is a matter of judgment anyway, and that the procedure outlined above is not needed. If a person will follow this procedure while gaining experience, and evaluate his or her results, the judgment developed should be more reliable than if developed without benefit of knowledge of the principles that are involved. The important point to remember is that the relative effects of normal random errors in any one direction are proportional to their *squares*.

Systematic errors are treated differently. Generally, an attempt is made to discover the errors and eliminate them or compensate for them. In the case of a position determined by three or more lines of position resulting from readings with constant error, the error might be eliminated by finding and applying that correction (including sign) which will bring all lines through a common point.

312. Mistakes

The recognition of a mistake, as contrasted with an error (Section 301), is not always easy, since a mistake may have any magnitude, and may be either positive or negative. A large mistake should be readily apparent if the navigator is alert and has an understanding of the size of error to be reasonably expected. A small mistake is usually not detected unless the work is checked.

If results by two methods are compared, as a dead reckoning position and a line of position, exact agreement is not to be expected. But if the discrepancy is unreasonably large, a mistake is logically suspected. The definition of "unreasonably large" is a matter of opinion. If the 99.9 percent areas of the two results just touch, it is *possible* that no mistake has been made. However, the *probability* of either one having so great an error is remote if the errors are normal. The probability of both having 99.9 percent error of opposite sign at the same instant is very small indeed. Perhaps a reasonable standard is that unless the most accurate result lies within the 95 percent area of the least accurate result, the possibility of a mistake should be investigated. Thus, if the areas of uncertainty shown in Figure 311a represent the 95 percent areas, it is probable that a mistake has been made.

As in other matters pertaining to navigation, judgment is important. The use to be made of the results is certainly a consideration. In the middle of an ocean passage a mistake is usually not serious, and will undoubtedly be corrected before it jeopardizes the safety of the vessel. But if landfall is soon to be made, or if search and rescue operations are to be based upon the position, almost any mistake is intolerable.

313. Conclusion

The correct identification of the nature of an error is important if the error is to be handled intelligently. Thus, the statement is sometimes made that a radio bearing need not be corrected if the receiver is within 50 miles of the transmitter.

The need for a correction arises from the fact that radio waves are assumed to follow great circles, and if radio bearings are to be plotted on a Mercator chart, the equivalent rhumb line is needed. The statement regarding 50 miles implies that the size of the correction is proportional to distance only. It overlooks the fact that latitude and direction of the bearing line are also important factors, and is therefore a dangerous statement unless its limitations are understood.

The recognition of the type of error is also important. A systematic error has quite a different effect than a random error, and cannot be reduced by additional readings unless some method or procedure is instituted which will cause the errors to cancel each other.

The errors for various percentage probabilities are usually of greater interest than the "average" value. The average of a large number of normal errors approaches zero, but the probable (50 percent) error might be quite large.

A person who understands the nature of errors avoids many pitfalls. Thus, the magnitude of the errors of individual lines of position is not a reliable indication of the size of the error of the fix obtained from them. The size of the ·triangle formed by three lines of position has often been used as a guide to the accuracy of the fix, although a large triangle might be the result of a large constant error if the objects observed are equally spaced in azimuth. On the other hand, two lines of position with small errors might produce a fix having a much larger error if the lines cross at a small angle.

314. References

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