CHAPTER 1

MATHEMATICS

ARITHMETIC

100. Definition

Arithmetic is that branch of mathematics dealing with computation by numbers. The principal processes involved are addition, subtraction, multiplication, and division. A number consisting of a single symbol (1, 2, 3, etc.) is **a digit**. Any number that can be stated or indicated, however large or small, is called a **finite number**; one too large to be stated or indicated is called an **infinite number**; and one too small to be stated or indicated is called an **infinitesimal number**.

The **sign** of a number is the indication of whether it is positive (+) or negative (-). This may sometimes be indicated in another way. Thus, latitude is usually indicated as *north* (N) or *south* (S), but if north is considered positive, south is then negative with respect to north. In navigation, the north or south designation of latitude and declination is often called the "name" of the latitude or declination. A **positive number** is one having a positive sign (+); a **negative number** is one having a negative sign (-). The **absolute value** of a number is that number without regard to sign. Thus, the absolute value of both (+) 8 and (-) 8 is 8. Generally, a number without a sign can be considered positive.

101. Significant Digits

Significant digits are those digits of a number which have a significance. Zeros at the left of the number and sometimes those at the right are excluded. Thus, 1,325, 1,001, 1.408, 0.00005926, 625.0, and 0.4009 have four significant digits each. But in the number 186,000 there may be three, four, five, or six significant digits depending upon the accuracy with which the number has been determined. If the quantity has only been determined to the nearest thousand then there are three significant digits, the zeros at the right not being counted. If the number has been determined to the nearest one hundred, there are four significant digits, the first zero at the right being counted. If the number has been determined to the nearest ten, there are five significant digits, the first two zeros on the right being counted. If the quantity has been determined to the nearest unit, there are six significant digits, the three zeros at the right being counted.

This ambiguity is sometimes avoided by expressing numbers in powers of 10. Thus, $18.6\ 10^4$ (18.6 x 10,000) indicates accuracy to the nearest thousand, $18.60\ 10^4$ to the nearest hundred, $18.600\ 10^4$ to the nearest ten, and $18.6000\ 10^4$ to the nearest unit. The position of the decimal is not important if the correct power of 10 is given. For example, $18.6\ 10^4$ is the same as $1.86\ 10^6$, $186\ 10^3$, etc. The small number above and to the right of 10 (the **exponent**) indicates the number of places the decimal point is to be moved to the right. If the exponent is negative, it indicates a reciprocal, and the decimal point is moved to the left. Thus, $1.86\ 10^{-6}$ is the same as 0.00000186. This system is called **scientific notation**.

102. Expressing Numbers

In navigation, fractions are usually expressed as decimals. Thus, $1/4$ is expressed as 0.25 and $1/3$ as 0.33. To determine the decimal equivalent of a fraction, divide the **numerator** (the number above the line) by the **denominator** (the number below the line). When a decimal is less than 1, as in the examples above, it is good practice to show the zero at the left of the decimal point (0.25, not .25).

A number should not be expressed using more significant digits than justified. The *implied* accuracy of a decimal is indicated by the number of digits shown to the right of the decimal point. Thus, the expression "14 miles" implies accuracy to the nearest whole mile, or any value between 13.5 and 14.5 miles. The expression "14.0 miles" implies accuracy of a tenth of a mile, or any value between 13.95 and 14.05 miles.

A quantity may be expressed to a greater implied accuracy than is justified by the accuracy of the information from which the quantity is derived. For instance, if a ship steams 1 mile in 3^m21^s, its speed is $60^m \div 3^m21^s = 60 \div 3.35 =$ 17.910447761194 knots, approximately. The division can be carried to as many places as desired, but if the time is measured only to the nearest second, the speed is accurate only to one decimal place in this example, because an error of 0.5 second introduces an error of more than 0.05 knot in the speed. Hence, the additional places are meaningless and possibly misleading, unless more accurate time is available. In general, it is not good practice to state a quantity to imply accuracy greater than what is justified. However, in marine navigation the accuracy of information is often unknown, and it is customary to give positions as if they were accurate to 0.1' of latitude and longitude, although they *may* not be accurate even to the nearest whole minute.

If there are no more significant digits, regardless of how far a computation is carried, this may be indicated by use of the word "exactly." Thus, 12÷4=3 exactly and 1 nautical mile = 1,852 meters exactly; but 12÷7=1.7 approximately, the word "approximately" indicating that additional decimal places might be computed. Another way of indicating an approximate relationship is by placing a positive or negative sign after the number. Thus, 12÷7=1.7+, and 11÷7=1.6-. This system has the advantage of showing whether the approximation is too great or too small.

In any arithmetical computation the answer is no more accurate than the least accurate value used. Thus, if it is desired to add 16.4 and 1.88, the answer might be given as 18.28, but since the first term might be anything from 16.35 to 16.45; the answer is anything from 18.23 to 18.33. Hence, to retain the second decimal place in the answer is to give a false indication of accuracy, for the number 18.28 indicates a value between 18.275 and 18.285. However, additional places are sometimes retained until the end of a computation to avoid an accumulation of small errors due to rounding off. In marine navigation it is customary to give most values to an accuracy of 0.1, even though some uncertainty may exist as to the accuracy of the last place. Examples are the dip and refraction corrections of sextant altitudes.

In general, a value obtained by interpolation in a table should not be expressed to more decimal places than given in the table.

Unless all numbers are exact, doubt exists as to the accuracy of the last digit in a computation. Thus, 12.3+9.4+4.6=26.3. But if the three terms to be added have been rounded off from 12.26, 9.38, and 4.57, the correct answer is 26.2, obtained by rounding off the answer of 26.21 found by retaining the second decimal place until the end. It is good practice to work with one more place than needed in the answer, when the information is available. In computations involving a large number of terms, or if greater accuracy is desired, it is sometimes advisable to retain two or more additional places until the end.

103. Rounding Off

In **rounding off** numbers to the number of places desired, one should take the nearest value. Thus, the number 6.5049 is rounded to 6.505, 6.50, 6.5, or 7, depending upon the number of places desired. If the number to be rounded off ends in 5, the nearer even number is taken. Thus, 1.55 and 1.65 are both rounded to 1.6. Likewise, 12.750 is rounded to 12.8 if only one decimal place is desired. However, 12.749 is rounded to 12.7. That is, 12.749 is not first rounded to 12.75 and then to 12.8, but the entire number is rounded in one operation. When a number ends in 5, the computation can sometimes be carried to additional places to determine whether the correct value is more or less than 5.

104. Reciprocals

The **reciprocal** of a number is 1 divided by that number. The reciprocal of a fraction is obtained by interchanging the numerator and denominator. Thus, the reciprocal of $3/5$ is $5/3$. A whole number may be considered a fraction with 1 as the denominator. Thus, 54 is the same as $54/1$, and its reciprocal is $1/54$. Division by a number produces the same result as multiplying by its reciprocal, or vice versa. Thus, $12\div2=12\times1/2=6$, and $12\times2=12\div1/2=24$.

105. Addition

When two or more numbers are to be added, it is generally most convenient to write them in a column, with the decimal points in line. Thus, if 31.2, 0.8874, and 168.14 are to be added, this may be indicated by means of the addition sign (+): 31.2+0.8874+168.14=200.2. But the addition can be performed more conveniently by arranging the numbers as follows:

$$\begin{array}{r@{.}l} 31&2 \\ 0&8874 \\ 168&14 \\ \hline 200&2 \end{array}$$

The answer is given only to the first decimal place, because the answer is no more accurate than the least precise number among those to be added, as indicated previously. Often it is preferable to state all numbers in a problem to the same precision before starting the addition, although this may introduce a small error:

$$\begin{array}{r} 31.2 \\ 0.9 \\ 168.1 \\ \hline 200.2 \end{array}$$

If there are no decimals, the last digit to the right is aligned:

$$\begin{array}{r} 166 \\ 2 \\ 96,758 \\ \hline 96,926 \end{array}$$

Numbers to be added should be given to the same absolute accuracy, when available, to avoid a false impression of accuracy in the result. Consider the following:

186, 000
71, 832
9, 614
728
268, 174

The answer would imply accuracy to six places. If the first number given is accurate to only three places, or to the nearest 1,000, the answer is not more accurate, and hence the answer should be given as 268,000. Approximately the same answer would be obtained by rounding off at the start:

186, 000
72, 000
10, 000
1, 000
269, 000

If numbers are **added arithmetically**, their absolute values are added without regard to signs; but if they are **added algebraically**, due regard is given to signs. If two numbers to be added algebraically have the same sign, their absolute values are added and given their common sign. If two numbers to be added algebraically have unlike signs, the smaller absolute value is subtracted from the larger, and the sign of the value having the larger absolute value is given to the result. Thus, if +8 and -7 are added arithmetically, the answer is 15, but if they are added algebraically, the answer is + 1.

An answer obtained by addition is called a **sum**.

106. Subtraction

Subtraction is the inverse of addition. Stated differently, the *addition* of a *negative* number is the same as the *subtraction* of a *positive* number. That is, if a number is to be subtracted from another, the sign (+ or -) of the **subtrahend** (the number to be subtracted) is reversed and the result added algebraically to the **minuend** (the number from which the subtrahend is to be subtracted). Thus, 6-4=2. This may be written +6-(+4)=+2, which yields the same result as +6+ (-4). For solution, larger numbers are often conveniently arranged in a column with decimal points in a vertical column, as in addition. Thus, 3,728.41-1,861.16 may be written:

(+)3,728.41
(+)1,861.16 (subtract)
(+)1,867.25

This is the same as:

(+)3,728.41
(-)1,861.16 (add algebraically)
(+)1,867.25

The rule of sign reversal applies likewise to negative numbers. Thus, if -3 is to be *subtracted* from +5, this may be written +5- (-3) =5+3=8. In the algebraic addition of two numbers of opposite sign (numerical subtraction), the smaller number is subtracted from the larger and the result is given the sign of the larger number. Thus, +7 -4= +3, and -7 +4= -3, which is the same as +4 -7=-3.

In navigation, numbers to be numerically subtracted are usually marked (-), and those to be numerically added are marked (+) or the sign is not indicated. However, when a sign is part of a designation, and the reverse process is to be used, the word "reversed" (rev.) is written after the number. Thus, if GMT is known and ZT in the (+) 5 zone is to be found (by subtraction), the problem may be written:

GMT	1754	
ZD	(+)5	(rev.)
ZT	1254	

The symbol ~ indicates that an absolute difference is required without regard to sign of the answer. Thus, 28~13=15, and 13~28=15. In both of these solutions 13 and 28 are positive and 15 is an absolute value without sign. If the signs or names of both numbers are the same, either positive or negative, the smaller is subtracted from the larger, but if they are of opposite sign or name, they are numerically added. Thus, (+)16~(+)21=5 and (-)16~(-)21=5, but (+)16~ (-)21=37 and (-)16~(+)21 =37. Similarly, the difference of latitude between 15°N and 20°N, or between 15°S and 20°S, is 5°, but the difference of latitude between 15°N and 20°S, or between15°S and 20°N, is 35°. If motion from one latitude to another is involved, the difference may be given a sign to indicate the direction of travel, or the location of one place with respect to another. Thus, if B is 50 miles west of A, and C is 125 miles west of A, B and C are 75 miles apart regardless of the direction of travel. However, B is 75 miles east of C, and C is 75 miles west of B. When direction is indicated, an algebraic difference is given, rather than an absolute difference, and the symbol ~ is not appropriate.

It is sometimes desirable to consider all addition and subtraction problems as addition, with negative signs (-) given before those numbers to be subtracted; so that there can be no question of which process is intended. The words "add" and "subtract" may be used instead of signs. In navigation, "names" (usually north, south, east, and west) are often used, and the relationship involved in a certain problem may need to be understood to determine whether to add or subtract. Thus, LHA=GHA-λ (west) and LHA=GHA+λ (east). This is the same as saying LHA=GHA-λ if west longitude is considered positive, for in this case, LHA=GHA-(-λ) or LHA=GHA+λ in east longitude, the same as before.

If numbers are **subtracted arithmetically**, they are subtracted without regard to sign; but if they are **subtracted**

algebraically, positive (+) numbers are *subtracted* and negative (-) numbers are *added*.

An answer obtained by subtraction is called a **difference**.

107. Multiplication

Multiplication may be indicated by the multiplication sign (×), as 154×28=4,312. For solution, the problem is conveniently arranged thus:

```
  154
(x)28
 1232
 308
 4312.
```

Either number may be given first, but it is generally more convenient to perform the multiplication if the larger number is placed on top, as shown. In this problem, 154 is first multiplied by 8 and then by 2. The second answer is placed under the first, but set one **place** to the left, so that the right-hand digit is directly below the 2 of the multiplier. These steps might be reversed, multiplication by 2 being performed first. This procedure is sometimes used in estimating.

When one number is placed below another for multiplication, as shown above, it is usually best to align the right-hand digits without regard for the position of the decimal point. The number of decimal places in the answer is the sum of the decimal places in the **multiplicand** (the number to be multiplied) and the **multiplier** (the second number):

```
   163.27
(x) 263.9
   146943
   48981
  97962
 32654
 43086.953
```

However, when a number ends in one or more zeros, these may be ignored until the end and then added on to the number:

```
  1924
 (x)1800
 15392
 1924
 3463200
```

This is also true if both multiplicand and multiplier end in zeros:

```
  1924000
 (x)1800
 15392
 1924
 3463200000
```

When negative values are to be multiplied, the sign of the answer is positive if an *even* number of negative signs appear, and negative if there are an *odd* number. Thus, 2×3=6, 2×(-3)=-6, -2×3=-6, -2× (-3)=(+)6. Also, 2×3×8× (-2)×5=-480, 2×(-3)×8×(-2)×5=480, 2×(-3)×(-8)×(-2) ×5=-480, 2×(-3)×(-8)×(-2)×(-5)=480, and (-2)×(-3)×(-8)×(-2)×(-5)=-480.

An answer obtained by multiplication is called a **product**. Any number multiplied by 1 is the number itself. Thus, 125×1=125. Any number multiplied by 0 is 0. Thus, 125×0=0 and l×0=0.

To multiply a number by itself is to **square** the number. This may be indicated by the **exponent** 2 placed to the right of the number and above the line as a **superior**. Thus, 15×15 may be written 15^2. Similarly, 15×I5×I5=15^3, and 15×15×15×15=15^4, etc. The exponent (2, 3, 4, etc.) indicates the **power** to which a number is to be **raised**, or how many times the number is to be used in multiplication. The expression 15^2 is usually read "15 squared", 15^3 is read "15 cubed" or "15 to the third power," 15^4 (or higher power) is read "15 to the fourth (or higher) power." The answer obtained by **raising to a power** is called the "square," "cube" etc., or the ... "power" of the number. Thus, 225 is the "square of 15", 3,375 is the "cube of 15" or the "third power of 15," etc. The zero power of any number except zero (if zero is considered a number) is 1. The zero power of zero is zero. Thus, 15^0=1 and 0^0=0.

Parentheses may be used to eliminate doubt as to what part of an expression is to be raised to a power. Thus, -32 may mean either -(3×3)=-9 or -3×-3=(+)9. To remove the ambiguity, the expression may be written -(3)2 if the first meaning is intended, and -$(3)^2$ if the second meaning is intended.

108. Division

Division is the inverse of multiplication. It may be indicated by the division sign (÷), as 376÷21=18 approximately; or by placing the number to be divided, called the **dividend** (376), over the other number, called the **divisor** (21), as $\frac{376}{21}$=18 approximately. The expression $\frac{376}{21}$ may be written 376/21 with the same meaning. Such a problem is conveniently arranged for solution as follows:

$$\begin{array}{rr} & 17 \\ 21 & \overline{)376} \\ & 21 \\ & \overline{166} \\ & \underline{147} \\ & 19. \end{array}$$

Since the **remainder** is 19, or more than half of the divisor (21), the answer is 18 to the nearest whole number.

An answer obtained by division is called a **quotient**. Any number divided by 1 is the number itself. Thus, 65÷1=65. A number cannot be divided by 0.

If the numbers involved are accurate only to the number of places given, the answer should not be carried to additional places. However, if the numbers are exact, the answer might be carried to as many decimal places as desired. Thus, 374÷21 =17.809523809523809523809523. . . When a series of digits repeat themselves with the same remainder, as 809523 (with remainder 17) in the example given above, an exact answer will not be obtained regardless of the number of places to which the division is carried. The series of dots (...) indicates a **repeating decimal**. In a non-repeating decimal, a plus sign (+) may be given to indicate a remainder, and a minus sign (-) to indicate that the last digit has been rounded to the next higher value. Thus, 18.68761 may be written 18.6876+ or 18.688-. If the last digit given is rounded off, the word “approximately” may be used instead of dots or a plus or minus sign.

If the divisor is a whole number, the decimal point in the quotient is directly above that of the dividend when the work form shown above is used. Thus, in the example given above, if the dividend had been 37.6 instead of 376, the quotient would have been 1.8 approximately. If the divisor is a decimal, both it and the dividend are multiplied by the power of 10 having an exponent equal to the number of decimal places in the divisor, and the division is then carried out as explained above. Thus, if there are two decimal places in the divisor, both divisor and dividend are multiplied by $10^2 = 100$. This is done by moving the decimal to the right until the divisor is a whole number. If necessary, zeros are added to the dividend. Thus, if 3.7 is to be divided by 2.11, both quantities are first multiplied by 10^2, and 370 is divided by 211. This is usually performed as follows:

$$\begin{array}{rl} & 1.75 \\ 2.11 & \overline{)3.7000} \\ & \underline{211} \\ & 1590 \\ & \underline{1477} \\ & 1130 \\ & \underline{1055} \\ & 75. \end{array}$$

If *both* the dividend and divisor are positive, or if *both* are negative, the quotient is positive; but if *either* is negative, the quotient is negative. Thus, 6÷3=2, (- 6)÷(-3)=+2, (-6)÷3=-2, and 6÷(-3)=-2.

The **square root** of a number is that number which, multiplied by itself, equals the given number. Thus, $15\times15=15^2=225$, and $\sqrt{225}=225^{1/2}=15$. The square root symbol $\sqrt{}$ is called the **radical sign**, or the exponent ½ indicates square root. Also, $\sqrt[3]{}$ or 1/3 as an exponent, indicates **cube root.** Fourth, fifth, or any root is indicated similarly, using the appropriate number. Nearly any arithmetic book explains the process of extracting roots, but this process is most easily performed by table, logarithms, or calculator. If no other means are available, it can be done by trial and error. The process of finding a root of a number is called **extracting a root.**

109. Logarithms

Though rarely used today, **logarithms** (“logs”) provide an easy way to multiply, divide, raise numbers to powers, and extract roots. The logarithm of a number is the power to which a fixed number, called the base, must be raised to produce the value to which the logarithm corresponds. The base of **common logarithm**, (given in tables 1 and 3) is 10. Hence, since $10^{1.8}=63$ approximately, 1.8 is the logarithm, approximately, of 63 to the base 10. In table 1 logarithms of numbers are given to five decimal **places**. This is sufficient for most purposes of the navigator. For greater precision, a table having additional places should be used. In general, the number of *significant digits* which are correct in an answer obtained by logarithms is the same as the number of *places* in the logarithms used.

A logarithm is composed of two parts. That part to the left of the decimal point is called the **characteristic**. That part to the right of the decimal point is called the **mantissa**. The principal advantage of using 10 as the base is that any given combination of digits has the same mantissa regardless of the position of the decimal point. Hence, only the mantissa is given in the main tabulation of table 1. Thus, the logarithm (mantissa) of 2,374 is given as 37548. This is correct for 2,374,000,000; 2,374; 23.74; 2.374; 0.2374; 0.000002374; or for any other position of the decimal point.

The position of the decimal point determines the characteristic, which is not affected by the actual digits involved. The characteristic of a whole number is one less than the number of digits. The characteristic of a **mixed decimal** (one greater than 1) is one less than the number of digits to the left of the decimal point. Thus, in the example given above, the characteristic of the logarithm of 2,374,000,000 is 9; that of 2,374 is 3; that of 23.74 is 1; and that of 2.374 is 0. The complete logarithms of these numbers are:

log 2,374,000,000	=9.37548
log 2,374	=3.37548
log 23.74	=1.37548
log 2.374	=0.37548

Since the mantissa of the logarithm of any multiple of ten is zero, the main table starts with 1,000. This can be considered 100, 10, 1, etc. Since the mantissa of these logarithms is zero, the logarithms consist of the characteristic only, and are whole numbers. Hence, the logarithm of 1 is 0 (0.00000), that of 10 is 1 (1.00000), that of 100 is 2 (2.00000), that of 1,000 is 3 (3.00000), etc.

The characteristic of the logarithm of a number less than 1 is negative. However, it is usually more conveniently indicated in a positive form, as follows: the characteristic is found by subtracting the number of zeros immediately to the right of the decimal point from 9 (or 19, 29, etc.) and following this by -10 (or -20, -30, etc.). Thus, the characteristic of the logarithm of 0.2374 is 9-10; that of 0.000002374 is 4-10; and that of 0:000000000002374 is 8-20. The complete logarithms of these numbers are:

log 2.374	=9.37548 - 10
log 0.000002374	=4.37548 - 10
log 0.000000000002374	=8.37548 - 20

When there is no question of the meaning, the -10 may be omitted. This is usually done when using logarithms of trigonometric functions, as shown in table 3. Thus, if there is no reasonable possibility of confusion, the logarithm of 0.2374 may be written 9.37548.

Occasionally, the logarithm of a number less than 1 is shown by giving the negative characteristic with a minus sign above it (since only the characteristic is negative, the mantissa being positive). Thus, the logarithms of the numbers given above might be shown thus:

$$\log 0.2374 = \bar{1}.37548$$

$$\log 0.000002374 = \bar{6}.37548$$

$$\log 0.000000000002374 = \overline{12}.37548$$

In each case, the negative characteristic is one more than the number of zeros immediately to the right of the decimal point.

There is no real logarithm of 0, since there is no *finite* power to which *any* number can be raised to produce 0. As numbers approach 0, their logarithms approach negative infinity.

To find the number corresponding to a given logarithm, called finding the **antilogarithm** (“antilog”); enter the table with the mantissa of the given logarithm and determine the corresponding number, interpolating if necessary. Locate the position of the decimal point by means of the characteristic of the logarithm, in accordance with the rules given above.

110. Multiplication by Logarithms

To *multiply* one number by another, *add* their logarithms and find the antilogarithm of the sum. Thus, to multiply 1,635.8 by 0.0362 by logarithms:

log 1635.8= 3.21373
log 0.0362= 8.55871-10 (add)
log 59.216=11.77244-10 or 1.77244

Thus, 1,635.8×0.0362=59.216. In navigation it is customary to use a slightly modified form, and to omit the -10 where there is no reasonable possibility of confusion, as follows:

1635.8	log 3.21373
0.0362	log 8.55871
59.216	log 1.77244

To *raise a number to a power*, multiply the logarithm of that number by the power indicated, and find the antilogarithm of the product. Thus, to find 13.156^3 by logarithms, using the navigational form:

13.156	log 1.11913	
x	3	(multiply)
2277.2	log 3.35739	

111. Division by Logarithms

To *divide* one number by another, subtract the logarithm of the divisor from that of the dividend, and find the antilogarithm of the remainder. Thus, to find 0.4637÷28.03 by logarithms, using the navigational form:

0.4637	log	9.66624	
28.03	log (-)	1.44762	(subtract)
0.016543	log	8.21862	

It is sometimes necessary to modify the first logarithm before the subtraction can be made. This would occur in the example given above, for instance, if the divisor and dividend were reversed, so that the problem became 28.03÷0.4637. In this case 10-10 would be added to the logarithm of the dividend, becoming 11.44762-10:

28.03	log	11.44762 - 10
0.4637	log (-)	9.66624 - 10
60.448	log	1.78138

One experienced in the use of logarithms usually carries this change mentally, without showing it in his or her work form:

28.03	log	1.44762
0.4637	log (-)	9.66624
60.448	log	1.78138

Any number can be added to the characteristic as long as that same number is also subtracted. Conversely, any number can be subtracted from the characteristic as long as that same number is also added.

To *extract a root* of a number, divide the logarithm of that number by the root indicated, and find the antilogarithm of the quotient. Thus, to find $\sqrt{7}$ by logarithms:

$$\begin{array}{rl} 7 & \log \underline{0.84510}\ (\div 2) \\ 2.6458 & \log 0.42255 \end{array}$$

To divide a negative logarithm by the root indicated, first modify the logarithm so that the quotient will have a -10. Thus, to find $\sqrt[3]{0.7}$ by logarithms:

$$\begin{array}{rl} 7 & \log \underline{29.84510} - 30\ (\div 3) \\ 0.88792 & \log 09.94837 - 10 \end{array}$$

or, carrying the -30 and -10 mentally,

$$\begin{array}{rl} 0.7 & \log \underline{29.84510}\ (\div 3) \\ 0.88792 & \log 9.94837 \end{array}$$

112. Cologarithms

The **cologarithm** ("colog") of a number is the value obtained by subtracting the logarithm of that number from zero, usually in the form 10-10. Thus, the logarithm of 18.615 is 1.26987. The cologarithm is:

$$\begin{array}{r} 10.00000-10 \\ (-)1.26987 \\ 8.73013 - 10 \end{array}$$

Similarly, the logarithm of 0.0018615 is 7.26987 -10, and its cologarithm is:

$$\begin{array}{r} 10.00000 - 10 \\ (-)\,\underline{7.26987 - 10} \\ 2.73013 \end{array}$$

The *cologarithm* of a number is the *logarithm* of the reciprocal of that number. Thus, the cologarithm of 2 is the logarithm of ½. Since division by a number is the same as multiplication by its reciprocal, the use of cologarithms permits division problems to be converted to problems of multiplication, eliminating the need for subtraction of logarithms. This is particularly useful when both multiplication and division are involved in the same problem. Thus, to find $\frac{92.732 \quad 0.0137 \quad 724.3}{0.516 \quad 3941.1}$ by logarithm, one might *add* the logarithms of the three numbers in the numerator, and *subtract* the logarithms of the two numbers in the denominator. If cologarithms are used for the numbers in the denominator, all logarithmic values are added. Thus, the solution might be made as follows:

92.732		log 1.96723
0.0137		log 8.13672
724.3		log 2.85992
0.516	log 9.71265	colog 0.28735
3941.1	log 3.59562	colog 6.40438
0.45248		log 9.65560

113. . Various Kinds of Logarithms

As indicated above, **common logarithms** use 10 as the base. These are also called **Brigg's logarithms**. For some purposes, it is convenient to use 2.7182818 approximately (designated e) as the base for logarithms. These are called **natural logarithms** or **Naperian logarithms** ($\log_e$). Common logarithms are shown as $\log_{10}$ when the base might otherwise be in doubt.

Addition and subtraction logarithms are logarithms of the sum and difference of two numbers. They are used when the logarithms of two numbers to be added or subtracted are known, making it unnecessary to find the numbers themselves.

114. Slide Rule

A **slide rule** is a mechanical analog computer. The slide rule is used primarily for multiplication and division, and also for functions such as roots, logarithms and trigonometry. The device is now obsolete with the advent of the hand held electronic calculator in the mid-1970's. Figure 114 depicts a typical slide rule.

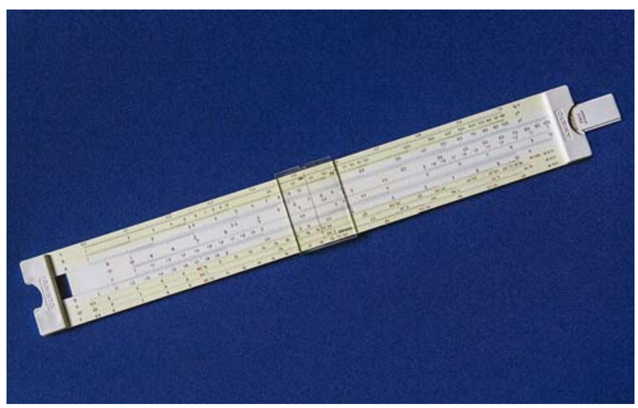

Figure 114. Slide rule. By Jan1959 (own work) via Wikimedia Commons

Slide rules come in many types and sizes, some designed for specific purposes. The most common form consists of an outer "body" or "frame" with grooves to permit a "slide" to be moved back and forth between the two outer parts, so that any graduation of a scale on the slide can be brought opposite any graduation of a scale on the body.

A cursor called an "indicator" or "runner" is provided to assist in aligning the desired graduations. In a **circular slide rule** the "slide" is an inner disk surrounded by a larger one, both pivoted at their common center. The scales of a slide rule are *logarithmic*. That is, they increase proportionally to the logarithms of the numbers indicated, rather than to the numbers themselves. This permits addition and subtraction of logarithms by simply measuring off part of the length of the slide from a graduated point on the body, or vice versa. Two or three complete scales within the length of the rule may be provided for finding squares, cubes, square roots, and cube roots.

Properly used, a slide rule can provide quick answers to many of the problems of navigation. However, its precision is usually limited to from two to four significant digits, and should not be used if greater precision is desired.

Great care should be used in placing the decimal point in an answer obtained by slide rule, as the correct location often is not immediately apparent. Its position is usually determined by making a very rough mental solution. Thus, 2.93×8.3 is *about* 3×8=24. Hence, when the answer by slide rule is determined to be "243," it is known that the correct value is 24.3, not 2.43 or 243.

115. Mental Arithmetic

Many of the problems of the navigator can be solved mentally. The following are a few examples.

If the speed is a number divisible into 60 a whole number of times, distance problems can be solved by a simple relationship. Thus, at 10 knots a ship steams 1 mile in $\frac{60}{10} = 6$ minutes. At 12 knots it requires 5 minutes, at 15 knots 4 minutes, etc. As an example of the use of such a relationship, a vessel steaming at 12 knots travels 5.6 miles in 28 minutes, since $\frac{28}{5}=5+\frac{3}{5}=5.6$, or 0.1 mile every half minute.

For relatively short distances, one nautical mile can be considered equal to 6,000 feet. Since one hour has 60 minutes, the speed in hundreds of feet per minute is equal to the speed in knots. Thus, a vessel steaming at 15 knots is moving at the rate of 1,500 feet per minute.

With respect to time, 6 minutes=0.1 hour, and 3 minutes = 0.05 hour. Hence, a ship steaming at 13 knots travels 3.9 miles in 18 minutes (13×0.3), and 5.8 miles in 27 minutes (13×0.45).

In arc units, 6' = 0.1° and 6" = 0.1'. This relationship is useful in rounding off values given in arc units. Thus, 17°23'44"=17°23.7' to the nearest 0.1', and 17.4° to the nearest 0.1°. A thorough knowledge of the six multiplication table is valuable. The 15 multiplication table is also useful, since $15° = 1^h$. Hence, $16^h = 16×15 = 240°$. This is particularly helpful in quick determination of zone description. Pencil and paper or a table should not be needed, for instance, to decide that a ship at sea in longitude 157°18.4' W is in the (+)10 zone.

It is also helpful to remember that $1° = 4^m$ and $1' = 4^s$. In converting the LMT of sunset to ZT, for instance, a quick mental solution can be made without reference to a table. Since this correction is usually desired only to the nearest whole minute, it is necessary only to multiply the longitude difference in degrees (to the nearest quarter degree) by four.

VECTORS

116. Scalars and Vector Quantities

A **scalar** is a quantity which has *magnitude* only; a **vector quantity** has both *magnitude* and *direction*. If a vessel is said to have a tank of 5,000 gallons capacity, the number 5,000 is a scalar. As used in this book, *speed* alone is considered a scalar, while *speed* and *direction* are considered to constitute *velocity*, a vector quantity. Thus, if a vessel is said to be steaming at 18 knots, without regard to direction, the number 18 is considered a scalar; but if the vessel is said to be steaming at 18 knots on course 157°, the combination of 18 knots, and 157° constitutes a vector quantity. *Distance* and *direction* also constitute a vector quantity.

A *scalar* can be represented fully by a number. A *vector quantity* vector requires, in addition, an indication of direction. This is conveniently done graphically by means of a straight line, the length of which indicates the *magnitude*, and the direction of which indicates the *direction* of application of the magnitude. Such a line is called a **vector**. Since a straight line has two directions, reciprocals of each other, an arrowhead is placed along or at one end of a vector to indicate the direction represented, unless this is apparent or indicated in some other manner.

117. Addition and Subtraction of Vectors

Two vectors can be *added* by *starting* the second at the termination (rather than the origin) of the first. A common navigational use of vectors is the dead reckoning plot of a vessel. Refer to Figure 117 depicting the addition and subtraction of vectors. If a ship starts at *A* and steams 18 miles on course 090° and then 12 miles on course 060°, it arrives by dead reckoning at *C*. The line *AB* is the vector for the first run, and *BC* is the vector for the second. Point *C* is the position found by *adding* vectors *AB* and *BC*. The vector *AC*, in this case the *course and distance made good*, is the **resultant**. Its value, both in direction and amount, can be determined by measurement. Lines *AB*, *BC*, and *AC* are all **distance vectors**. **Velocity vectors** are used when determining the effect of, or allowing for, current, interconverting true and apparent wind, and solving relative motion problems.

The **reciprocal** of a vector has the same magnitude but opposite direction of the vector. To *subtract* a vector, *add* it's reciprocal. This is indicated by the broken lines in Figure 117, in which the vector *BC*' is drawn in the opposite direction to *BC*. In this case the resultant is *AC*'. Subtraction of vectors is involved in some current and wind problems.

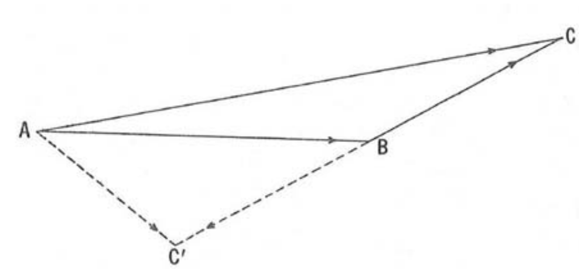

Figure 117. Addition and subtraction of vectors.

ALGEBRA

118. Definitions

Algebra is that branch of mathematics dealing with computation by letters and symbols. It permits the mathematical statement of certain relationships between variables. When numbers are substituted for the letters, algebra becomes arithmetic. Thus if $a=2b$, any value may be assigned to b, and a can be found by multiplying the assigned value by 2. Any statement of equality (as $a=2b$) is an **equation**. Any combination of numbers, letters, and symbols (as $2b$) is a **mathematical expression**.

119. Symbols

As in arithmetic, plus (+) and minus (-) signs are used, and with the same meaning. Multiplication (×) and division (÷) signs are seldom used. In algebra, $a \times b$ is usually written ab, or sometimes a·b. For division $a \div b$ is usually written $\frac{a}{b}$ or a/b. The symbol > means "greater than" and < means "less than." Thus, $a > b$ means "a is greater than b," and $a \geq b$ means "a is equal to or greater than b."

The order of performing the operations indicated in an equation should be observed carefully. Consider the equation $a=b+cd-e/f$. If the equation is to be solved for a, the value cd should be determined by multiplication and e/f by division *before* the addition and subtraction, as each of these is to be considered a single quantity in making the addition and subtraction. Thus, if $cd=g$ and $e/f=h$, the formula can be written $a=b+g-h$.

If an equation including both multiplication and division between plus or minus signs is not carefully written, some doubt may arise as to which process to perform first. Thus, $a \div b \times c$ or $a/b \times c$ may be interpreted to mean either that a/b is to be multiplied by c or that a is to be divided by $b \times c$. Such an equation is better written ac/b if the first meaning is intended, or a/bc if the second meaning is intended.

Parentheses, ( ), may be used for the same purpose or to indicate any group of quantities that is to be considered a single quantity. Thus, a(b+c) is an indication that the sum of b and c is to be multiplied by a. Similarly, a+(b-c)2 indicates that c is first to be subtracted from b, and then the result is to be squared and the value thus obtained added to a. When an expression within parentheses is part of a larger expression which should also be in parentheses, brackets, [ ], are used in place of the outer parentheses. If yet another set is needed, braces, { }, are used.

A quantity written $\sqrt{3}\,ab$ is better written $ab\,\sqrt{3}$ to remove any suggestion that the square root of $3ab$ is to be found.

120. Addition and Subtraction

Addition and subtraction.-A plus sign before an expression in parentheses means that each term retains its sign as given. Thus, $a + (b + c - d)$ is the same as $a + b + c - d$. A minus sign preceding the parentheses means that each sign within the parentheses is to be reversed. For example, $a - (b + c - d) = a - b - c + d$.

In any equation involving addition and subtraction, similar terms can be combined. Thus, $a+b+c+b-2c-d=a+2b-c-d$. Also, $a+3ab+a^2-b-ab=a+2ab+a^2-b$. That is, to be combined, the terms must be truly alike, for a cannot be combined with ab, or with a^2.

Equal quantities can be added to or subtracted from both members of an equation without disturbing the equality. Thus, if $a=b$, $a+2=b+2$, or $a+x=b+x$. If $x=y$, then $a+x=b+y$.

121. Multiplication and Division

When an expression in parentheses is to be multiplied by a quantity outside the parentheses, each quantity separated by a plus or minus sign within the parentheses should be multiplied separately. Thus, $a(b+cd-e/f)$ may be written $ab+acd-ae/f$. Any quantity appearing in *every* term of one member of an equation can be separated out by **factoring**, or dividing each term by the common quantity. Thus, if $a = bc + \frac{bd}{e} - b^2 + b$, the equation may be written

$a = b\left(c + \frac{d}{e} - b + 1\right)$.

Note that $\frac{b}{b}$ = and $\frac{b^2}{b} = b$. This is the inverse of multiplication: $a \times 1 = a$, but $a \times a = a^2$. Also, $a^2 \times a^3 = a^5$; and $\frac{a^7}{a^2} = a^5$. Thus, in multiplying a power of a number by a power of the same number, the powers are added, or, stated mathematically, a^m x $a^n = a^{m+n}$. In division, $\frac{a^m}{a^n} = a^{m-n}$, or the exponents are subtracted. If n is greater than m, a *negative* exponent results. A value with a negative exponent is equal to the reciprocal of the same value with a positive exponent. Thus, $a^{-n} = \frac{1}{a^n}$ and $\frac{a^2b^{-3}}{c} = \frac{a^2}{b^3c}$.

In raising to a power a number with an exponent, the two exponents are multiplied. Thus, $(a^2)^3 = a^{2x3} = a^6$, or $(a^n)^m = a^{nm}$. The inverse is true in extracting a root. Thus, $\sqrt[3]{a^2} = a^{\frac{2}{3}} = a^{0.667}$, or $\sqrt[m]{a^n} = a^{\frac{n}{m}}$.

Both members of an equation can be multiplied or divided by equal quantities without disturbing the equality, excluding division by zero or some expression equal to zero. Thus, if $a=b+c$, $2a=2(b+c)$, or if $x=y$, $ax=y(b+c)$ and $\frac{a}{x} = \frac{b+c}{y}$. Sometimes there is more than one answer to an equation. Division by one of the unknowns may eliminate one of the answers.

Both members of an equation can be raised to the same power, and like roots of both members can be taken, without disturbing the equality. Thus, if $a=b+c$, $a^2=(b+c)^2$, or if $x=y$, $a^x=(b+c)^y$. This is *not* the same as $a^x=b^y+c^y$. Similarly, if $a=b+c$, $\sqrt{a} = \sqrt{b+c}$, or if $x=y$, $\sqrt[x]{a} = \sqrt[y]{b+c}$. Again, $\sqrt[v]{b+c}$ is *not* equal to $\sqrt[v]{b} + \sqrt[v]{c}$, as a numerical example will indicate: $\sqrt{100} = \sqrt{64+36}$, but $\sqrt{100}$ does not equal $\sqrt{64} + \sqrt{36}$.

If two quantities to be multiplied or divided are both positive or both negative, the result is positive. Thus, $(+a)$x$(+b)=ab$ and $\frac{-a}{-b} = +\frac{a}{b}$. But if, the signs are opposite, the answer is negative. Thus, $(+a)$x$(-b)=-ab$, and $\frac{-a}{+b} = -\frac{a}{b}$; also, $(-a)$x$(+b)$=-ab, and $\frac{+a}{-b} = -\frac{a}{b}$.

In expressions containing both parentheses and brackets, or both of these and braces, the innermost symbols are removed first. Thus, $-\left\{6z - \frac{x(x+4) - 5y}{y}\right\} = -\left\{6z - \frac{[x^2 + 4x - 5y]}{y}\right\} = -\left\{6z - \frac{x^2}{y} - \frac{4x}{y} + 5\right\} = -6z + \frac{x^2}{y} + \frac{4x}{y} - 5.$

122. Fractions

To add or subtract two or more fractions, convert each to an expression having the same denominator, and then add the numerators.

Thus, $\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf}{bdf} + \frac{cbf}{bdf} + \frac{ebd}{bdf} = \frac{adf + cbf + ebd}{bdf}$. That is, both numerator and denominator of each fraction are multiplied by the denominator of the other remaining fractions.

To multiply two or more fractions, multiply the numerators by each other, and also multiply the denominators by each other. Thus, $\frac{a}{b} \quad \frac{c}{d} \quad \frac{e}{f} = \frac{ace}{bdf}$.

To divide two fractions, invert the divisor and multiply. Thus, $\frac{a}{b} \quad \frac{c}{d} = \frac{a}{b} \quad \frac{d}{c} = \frac{ad}{bc}$.

If the same factor appears in all terms of a fraction, it can be factored out without changing the value of the fraction. Thus, $\frac{ab + ac + ad}{ae - af} = \frac{b + c + d}{e - f}$. This is the same as factoring a from the numerator and denominator separately. That is, $\frac{ab + ac + ad}{ae - af} = \frac{a(b+c+d)}{a(e-f)}$, but since $\frac{a}{a}$ =1, this part can be removed, and the fraction appears as above.

123. Transposition

It is sometimes desirable to move terms of an expression from one side of the equals sign (=) to the other. This is called **transposition**, and to move one term is to **transpose** it. If the term to be moved is preceded by a plus or a minus sign, this sign is reversed when the term is transposed. Thus, if $a = b + c$, then $a - b = c$, $a - c = b$, $-b = c - a$, $-b - c = -a$, $-b - c = -a$, etc. Note that the signs of all terms can be reversed without destroying the equality, for if $a = b$, $b = a$. Thus, if all terms to the left of the equals sign are exchanged for all those to the right, no change in sign need take place, yet if each is moved individually, the signs reverse. For instance, if $a = b + c$, $-b - c = -a$. If each term is multiplied by -1, this becomes $b + c = a$.

A term which is to be multiplied or divided by *all* other terms on its side of the equation can be transposed if it is also moved from the numerator to the denominator, or vice versa. Thus, if $a = \frac{b}{c}$, then $ac = b$, $c = \frac{b}{a}$, $\frac{1}{b} = \frac{1}{ac}$, $\frac{c}{b} = \frac{1}{a}$, etc. (Note that $a = \frac{a}{1}$.) The same result could be obtained by multiplying both sides of an equation by the same quantity. For instance, if both sides of $a = \frac{b}{c}$ are multiplied by c, the equation becomes $ac = \frac{bc}{c}$ and since any number (except zero) divided by itself is unity, $\frac{c}{c} = 1$, and the equation becomes $ac = b$, as given above. Note, also, that *both* sides of an equation can be *inverted* without destroying the relationship, for if $a = b$, $\frac{a}{1} = \frac{b}{1}$, and $\frac{1}{b} = \frac{1}{a}$ or $\frac{1}{a} = \frac{1}{b}$. This is accomplished by transposing *all* terms of an equation.

Note that in the case of transposition by changing the plus or minus sign, an entire expression must be changed, and not a part of it. Thus, if $a = bc + d$, $a - bc = d$, but it is not true that $a + b = c + d$. Similarly, a term to be transposed by reversing its multiplication-division relationship must bear that relationship to *all* other terms on its side of the equation. That is, if $a = bc + d$, it is *not* true that $\frac{a}{b} = c + d$, or that $\frac{a}{bc} = d$, but $\frac{a}{bc + d} = 1$. if $a = b(cd + e)$.

124. Ratio and Proportions

If the relationship of a to b is the same as that of c to d, this fact can be written $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$. Either side of this equation, $\frac{a}{b}$ or $\frac{c}{d}$ is called a **ratio** and the whole equation is called a **proportion**. When a ratio is given a numerical value, it is often expressed as a decimal or as a percentage. Thus, if $\frac{a}{b} = \frac{1}{4}$ (that is, $a = 1$, $b = 4$), the ratio might be expressed as 0.25 or as 25 percent.

Since a ratio is a fraction, it can be handled as any other fraction.

GEOMETRY

125. Definition

Geometry deals with the properties, relations, and measurement of lines, surfaces, solids, and angles. **Plane geometry** deals with plane figures, and **solid geometry** deals with three–dimensional figures.

A **point**, considered mathematically, is a place having position but no extent. It has no length, breadth, or thickness. A point in motion produces a **line**, which has length, but neither breadth nor thickness. A **straight or right line** is the shortest distance between two points in space. A line in motion in any direction except along itself produces a **surface**, which has length and breadth, but not thickness. A **plane surface** or **plane** is a surface without curvature. A straight line connecting any two of its points lies wholly within the plane. A plane surface in motion in any direction except within its plane produces a **solid**, which has length, breadth, and thickness. **Parallel** lines or surfaces are those which are everywhere equidistant. **Perpendicular** lines or surfaces are those which meet at right or 90° angles. A perpendicular may be called a **normal**, particularly when it is perpendicular to the tangent to a curved line or surface at the point of tangency. All points equidistant from the ends of a straight line are on the perpendicular bisector of that line. The shortest distance from a point to a line is the length of the perpendicular between them.

126. Angles

An **angle** is formed by two straight lines which meet at a point. It is measured by the arc of a circle intercepted between the two lines forming the angle, the center of the circle being at the point of intersection. In Figure 126a, the angle formed by lines *AB* and *BC*, may be designated "angle *B*," "angle *ABC*," or "angle *CBA*"; or by Greek letter as "angle α." The three letter designation is preferred if there is more than one angle at the point. When three letters are used, the middle one should always be that at the **vertex** of the angle.

An **acute angle** is one less than a right angle (90°).

A **right angle** is one whose sides are perpendicular (90°).

An **obtuse angle** is one greater than a right angle (90°) but less than 180°.

A **straight angle** is one whose sides form a continuous straight line (180°).

A **reflex angle** is one greater than a straight angle (180°) but less than a circle (360°). Any two lines meeting at a point form two angles, one less than a straight angle of 180° (unless exactly a straight angle) and the other greater than a straight angle.

An **oblique angle** is any angle not a multiple of 90°.

Two angles whose sum is a right angle (90°) are **complementary angles**, and either is the **complement** of the other.

Two angles whose sum is a straight angle (180°) are **supplementary angles**, and either is the **supplement** of the other.

Two angles whose sum is a circle (360°) are **explementary angles**, and either is the **explement** of the other. The two angles formed when any two lines terminate at a common point are explementary.

If the sides of one angle are perpendicular to those of another, the two angles are either equal or supplementary. Also, if the sides of one angle are parallel to those of anoth-

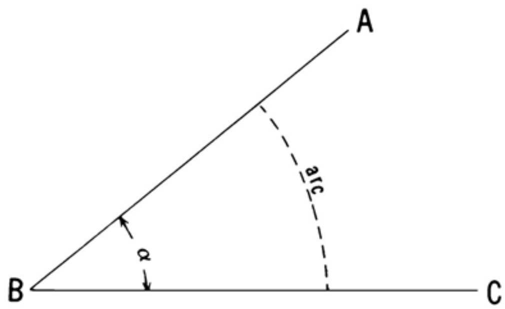

Figure 126a. Acute, right, and obtuse angles.

er, the two angles are either equal or supplementary.

When two straight lines intersect, forming four angles, the two opposite angles, called **vertical angles**, are equal. Angles which have the same vertex and lie on opposite sides of a common side are **adjacent angles**. Adjacent angles formed by intersecting lines are supplementary, since each pair of adjacent angles forms a straight angle. Thus, in Figure 126a, lines *AE* and BF intersect at *G*. Angles *AGB* and *EGF* form a pair of equal acute vertical angles, and *BGE* and *AGF* form a pair of equal obtuse vertical angles.

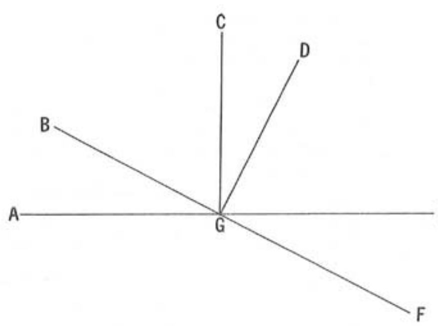

Figure 126b. Acute, right, and obtuse angles.

A **transversal** is a line that intersects two or more other lines. If two or more parallel lines are cut by a transversal, groups of adjacent and vertical angles are formed, as shown in Figure 126b. In this situation, all acute angles (*A*) are equal, all obtuse angles (*B*) are equal, and each acute angle is supplementary to each obtuse angle.

A **dihedral angle** is the angle between two intersecting planes.

127. Triangles

A **plane triangle** is a closed figure formed by three straight lines, called **sides**, which meet at three points called **vertices**. The vertices are labeled with capital letters and the sides with lowercase letters, as shown in Figure 127a, which depicts a triangle.

An **equilateral triangle** is one with its three sides equal in length. It must also be **equiangular**, with its three

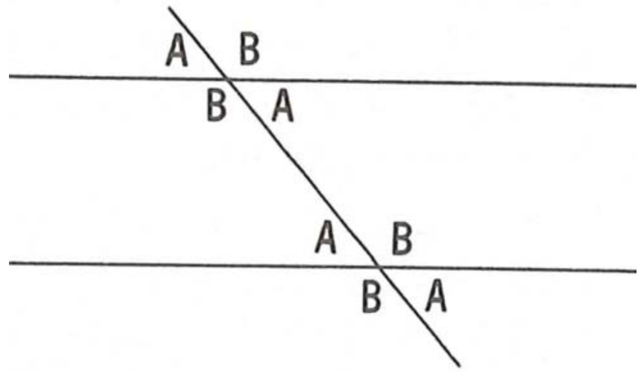

Figure 126c. Angles formed by a transversal.

angles equal.

An **isosceles triangle** is one with two equal sides, called **legs**. The angles opposite the legs are equal. A line which **bisects** (divides into two equal parts) the unequal angle of an isosceles triangle is the perpendicular bisector of the opposite side, and divides the triangle into two equal right triangles.

A **scalene triangle** is one with no two sides equal. In such a triangle, no two angles are equal.

An **acute triangle** is one with three acute angles.

A **right triangle** is one having a right angle. The side opposite the right angle is called the **hypotenuse**. The other two sides may be called **legs**. A plane triangle can have only one right angle.

An **obtuse triangle** is one with an obtuse angle. A plane triangle can have only one obtuse angle.

An **oblique triangle** is one which does not contain a right angle.

The **altitude** of a triangle is a line or the distance from any vertex perpendicular to the opposite side.

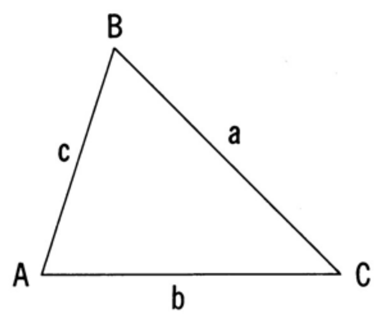

Figure 127a. A triangle.

A **median** of a triangle is a line from any vertex to the center of the opposite side. The three medians of a triangle meet at a point called the **centroid** of the triangle. This point divides each median into two parts, that part between the centroid and the vertex being twice as long as the other part.

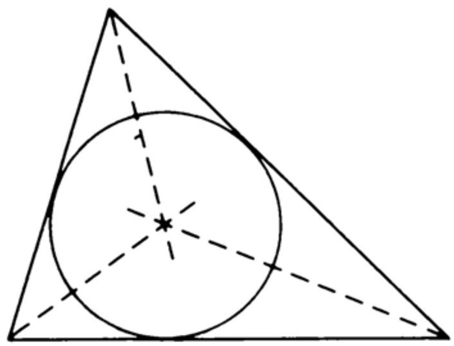

Figure 127b. A circle inscribed in a triangle.

Lines bisecting the three *angles* of a triangle meet at a point which is equidistant from the three sides, which is the center of the **inscribed circle**, as shown in Figure 127b. This point is of particular interest to navigators because it is the point theoretically taken as the fix when three lines of position of equal weight and having only random errors do not meet at a common point. In practical navigation, the point is found visually, not by construction, and other factors often influence the chosen fix position.

The perpendicular bisectors of the three *sides* of a triangle meet at a point which is equidistant from the three vertices, which is the center of the **circumscribed circle**, the circle through the three vertices and the smallest circle which can be drawn enclosing the triangle. The center of a circumscribed circle is *within* an acute triangle, *on the hypotenuse* of a right triangle, and *outside* an obtuse triangle.

A line connecting the mid–points of two sides of a triangle is always parallel to the third side and half as long. Also, a line parallel to one side of a triangle and intersecting the other two sides divides these sides proportionally. This principle can be used to divide a line into any number of equal or proportional parts. Refer to Figure 127c, which depicts dividing a line into equal parts. Suppose it is desired to divide line *AB* into four equal parts. From *A* draw any line *AC*. Along *C* measure four equal parts of any convenient lengths (*AD*, *DE*, *EF*, and *FG*). Draw *GB*, and through *F*, *E*, and *D* draw lines parallel to *GB* and intersecting *AB*. Then *AD'*, *D' E'*, *E' F'*, and *F' B* are equal and *AB* is divided into four equal parts.

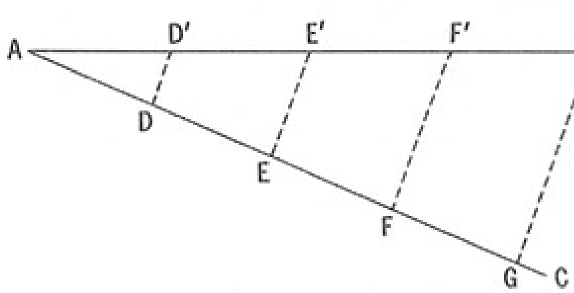

Figure 127c. Dividing a line into equal parts.

The sum of the angles of a plane triangle is always 180°. Therefore, the sum of the acute angles of a right triangle is 90°, and the angles are complementary. If one side of a triangle is extended, the **exterior angle** thus formed is supplementary to the adjacent **interior angle** and is therefore equal to the sum of the two non adjacent angles. If two angles of one triangle are equal to two angles of another triangle, the third angles are also equal, and the triangles are **similar**. If the area of one triangle is equal to the area of another, the triangles are **equal**. Triangles having equal bases and altitudes also have equal areas. Two figures are **congruent** if one can be placed over the other to make an exact fit. Congruent figures are both similar and equal. If any side of one triangle is equal to any side of a similar triangle, the triangles are congruent. For example, if two right triangles have equal sides, they are congruent; if two right triangles have two corresponding sides equal, they are congruent. Triangles are congruent only if the sides and angles are equal.

The sum of two sides of a plane triangle is always greater than the third side; their difference is always less than the third side.

The area of a triangle is equal to 1/2 of the area of the polygon formed from its base and height. If A = area, b = one of the legs of a right triangle or the base of any plane triangle, h = altitude, c = the hypotenuse of a right triangle, a = the other leg of a right triangle, and S = the sum of the interior angles:

$$\text{Area of plane triangle } A = \frac{bh}{2}$$

Sum of interior angles of plane triangle: S = 180°

The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides, or $a^2 + b^2 = c^2$. Therefore the length of the hypotenuse of plane right triangle can be found by the formula:

$$c = \sqrt{a^2 + b^2}$$

128. Polygons

A **polygon** is a closed plane figure made up of three or more straight lines called **sides**. A polygon with three sides is a **triangle**, one with four sides is a **quadrilateral**, one with five sides is a **pentagon**, one with six sides is a **hexagon**, and one with eight sides is an **octagon**. An **equilateral polygon** has equal sides. An **equiangular polygon** has equal interior angles. A **regular polygon** is both equilateral and equiangular. As the number of sides of a regular polygon increases, the figure approaches a circle.

A **trapezoid** is a quadrilateral with one pair of opposite sides parallel and the other pair not parallel. A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel. Any side of a parallelogram, or either of the parallel sides of a trapezoid, is the **base** of the figure. The perpendicular distance from the base to the opposite side is the altitude. A **rectangle** is a parallelogram with four right angles. (If anyone is a right angle, the other three must be, also.) A **square** is a rectangle with equal sides. A **rhomboid** is a parallelogram with oblique angles. A **rhombus** is a rhomboid with equal sides.

The sum of the exterior angles of a convex polygon (one having no interior reflex angles), made by extending each side in one direction only (consistently), is 360°.

A **diagonal** of a polygon is a straight line connecting any two vertices which are not adjacent. The diagonals of a parallelogram bisect each other.

The **perimeter** of a polygon is the sum of the lengths of its sides.

If A = area, s = the side of a square, a = that side of a rectangle adjacent to the base or that side of a trapezoid parallel to the base, b = the base of a quadrilateral, h = the altitude of a parallelogram or trapezoid, S = the sum of the angles of a polygon, and n = the number of sides of a polygon:

Area of a square: $A = s^2$

Area if a rectangle: $A = ab$

Area of a parallelogram: $A = bh$

Area of a trapezoid: $A = \frac{(a+b)h}{2}$

Sum of angles in convex polygon: $S = (n-2)180°$.

129. Circles

A **circle** is a plane, closed curve, all points of which are equidistant from a point within, called the **center**. See Figure 129 depicting elements of a circle.

The distance around a circle is called the **circumference**. Technically the length of this line is the **perimeter**, although the term "circumference" is often used. An arc is part of a circumference. A **major arc** is more than a semicircle (180°), a **minor arc** is less than a semicircle (180°). A **semi–circle** is half a circle (180°), a **quadrant** is a quarter of a circle (90°), a **quintant** is a fifth of a circle (72°), a **sextant** is a sixth of a circle (60°), an **octant** is an eighth of a circle (45°). Some of these names have been applied to instruments used by navigators for measuring altitudes of celestial bodies because of the part of a circle used for the length of the arc of the instrument.

Concentric circles have a common center. A **radius** (plural **radii**) or **semidiameter** is a straight line connecting the center of a circle with any point on its circumference. In Figure 129, *CA*, *CB*, *CD*, and *CE* are radii

A **diameter** of a circle is a straight line passing through its center and terminating at opposite sides of the circumference, or two radii in opposite directions (*BCD*, Figure 129). It divides a circle into two equal parts. The ratio of the length of the circumference of any circle to the length of its

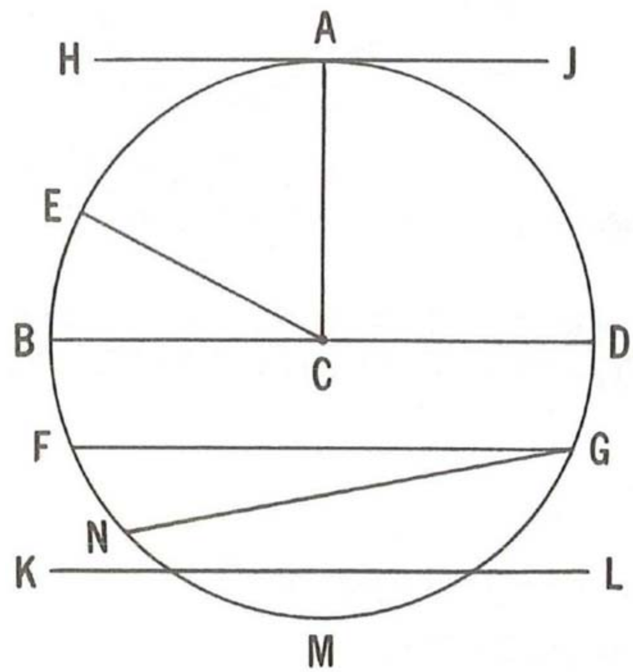

Figure 129. Elements of a circle.

diameter is 3.14159+, or π (the Greek letter pi), a relationship that has many useful applications.

A **sector** is that part of a circle bounded by two radii and an arc. In Figure 129, *BCE*, *ECA*, *ACD*, *BCA*, and *ECD* are sectors.The angle formed by two radii is called a **central angle**. Any pair of radii divides a circle into sectors, one less than a semicircle (180°) and the other greater than a semicircle (unless the two radii form a diameter).

A **chord** is a straight line connecting any two points on the circumference of a circle (*FG*, *GN* in Figure 129). Chords equidistant from the center of a circle are equal in length.

A **segment** is the part of a circle bounded by a chord and the intercepted arc (*FGMF*, *NGMN* in Figure 129). A chord divides a circle into two segments, one less than a semicircle (180°), and the other greater than a semicircle (unless the chord is a diameter). A diameter perpendicular to a chord bisects it, its arc, and its segments. Either pair of vertical angles formed by intersecting chords has a combined number of degrees equal to the sum of the number of degrees in the two arcs intercepted by the two angles.

An **inscribed angle** is one whose vertex is on the circumference of a circle and whose sides are chords (*FGN* in Figure 129). It has half as many degrees as the arc it intercepts. Hence, an angle inscribed in a semicircle is a right angle if its sides terminate at the ends of the diameter forming the semicircle.

A **secant** of a circle is a line intersecting the circle, or a chord extended beyond the circumference (*KL* in Figure 129).

A **tangent** to a circle is a straight line, in the plane of the circle, which has only one point in common with the circumference (*HJ* in Figure 129). A tangent is perpendicular to the radius at the **point of tangency** (A in Figure 129). Two tangents from a common point to opposite sides of a circle are equal in length, and a line from the point to the center of the circle bisects the angle formed by the two tangents. An angle formed outside a circle by the intersection of two tangents, a tangent and a secant, or two secants has *half* as many degrees as the *difference* between the two intercepted arcs. An angle formed by a tangent and a chord, with the apex at the point of tangency, has half as many degrees as the arc it intercepts. A **common tangent** is one tangent to more than one circle. Two circles are tangent to each other if they touch at one point only. If of different sizes, the smaller circle may be either inside or outside the larger one.

Parallel lines intersecting a circle intercept equal arcs.

If A = area; r = radius; d = diameter; C = circumference; s = linear length of an arc; a = angular length of an arc, or the angle it subtends at the center of a circle, in degrees; b = angular length of an arc, or the angle it subtends at the center of a circle, in radians; *rad* = radians and *sin* = sine:

$$\text{Circumference of a circle } C = 2\pi r = \pi d = 2\pi \text{ rad}$$

$$\text{Area of circle } A = \pi r^2 = \frac{\pi d^2}{4}$$

$$\text{Area of sector} = \frac{\pi r^2 a}{360} = \frac{r^2 b}{2} = \frac{rs}{2}$$

$$\text{Area of segment} = \frac{r^2(b - \sin a)}{2}$$

130. Polyhedrons

A **polyhedron** is a solid having plane sides or faces.

A **cube** is a polyhedron having six square sides.

A **prism** is a solid having parallel, similar, equal, plane geometric figures as bases, and parallelograms as sides. By extension, the term is also applied to a similar solid having nonparallel bases, and trapezoids or a combination of trapezoids and parallelograms as sides. The **axis** of a prism is the straight line connecting the centers of its bases. A **right prism** is one having bases perpendicular to the axis. The sides of a right prism are rectangles. A **regular prism** is a right prism having regular polygons as bases. The **altitude** of a prism is the perpendicular distance between the planes of its bases. In the case of a right prism it is measured along the axis.

A **pyramid** is a polyhedron having a polygon as one end, the **base**; and a point, the **apex**, as the other; the two ends being connected by a number of triangular sides or **faces**. The **axis** of a pyramid is the straight line connecting the apex and the center of the base. A **right pyramid** is one having its base perpendicular to its axis. A **regular pyramid** is a right pyramid having a regular polygon as its base. The altitude of a pyramid is the perpendicular distance from its apex to the plane of its base. A **truncated pyramid** is that portion of a pyramid between its base and a plane intersecting all of the faces of the pyramid.

If A = area, s = edge of a cube or slant height of a regular pyramid (from the center of one side of its base to the apex), V = volume, a = side of a polygon, h = altitude, P = perimeter of base, n = number of sides of polygon, B = area of base, and r = perpendicular distance from the center of side of a polygon to the center of the polygon:

Cube:

Area of each face: $A = s^2$

Total area of all faces: $A = 6s^2$

Volume: $V = s^3$

Regular prism:

Area of each face: $A = ah$

Total area of all faces: $A = Ph = nah$

Area of each base: $B = \frac{nar}{2}$

Total area of both bases: $A = nar$

Volume: $V = Bh = \frac{narh}{2}$

Regular pyramid:

Area of each face: $A = \frac{as}{2}$

Total area of all faces: $A = \frac{nas}{2}$

Area of base: $B = \frac{nar}{2}$

Volume: $V = \frac{Bh}{3} = \frac{narh}{6}$

131. Cylinders

A **cylinder** is a solid having two parallel plane **bases** bounded by closed congruent curves, and a surface formed by an infinite number of parallel lines, called **elements**, connecting similar points on the two curves. A cylinder is similar to a prism, but with a curved lateral surface, instead of a number of flat sides connecting the bases. The axis of a cylinder is the straight line connecting the centers of the bases. A **right cylinder** is one having bases perpendicular to the axis. A **circular cylinder** is one having circular bases. The **altitude** of a cylinder is the perpendicular distance between the planes of its bases. The **perimeter** of a base is the length of the curve bounding it.

If A = area, P = perimeter of base, h = altitude, r = radius of a circular base, B = area of base, and V = volume, then for a right circular cylinder:

Lateral area: $A = Ph = 2\pi rh$

Area of each base: $B = \pi r^2$

Total area, both bases: $A = 2\pi r^2$

Volume: $V = Bh = \pi r^2 h$

132. Cones

A **cone** is a solid having a plane **base** bounded by a closed curve, and a surface formed by lines, called **elements**, from every point on the curve to a common point called the **apex**. A cone is similar to a pyramid, but with a curved surface connecting the base and apex, instead of a number of flat sides. The **axis** of a cone is the straight line connecting the apex and the center of the base. A **right cone** is one having its base perpendicular to its axis. A **circular cone** is one having a circular base. The **altitude** of a cone is the perpendicular distance from its apex to the plane of its base. A **frustum** of a cone is that portion of the cone between its base and any parallel plane intersecting all elements of the cone. A **truncated cone** is that portion of a cone between its base and any nonparallel plane which intersects all elements of the cone but does not intersect the base.

If A = area, r = radius of base, s = slant height or length of element, B = area of base, h = altitude, and V = volume, then for a right circular cone:

Lateral area: $A = \pi rs$

Area of base: $B = \pi r^2$

Slant height: $s = \sqrt{r^2 + h^2}$

Volume: $V = \frac{Bh}{3} = \frac{\pi r^2 h}{3}$

133. Conic Sections

If a right circular cone of indefinite extent is intersected by a plane perpendicular to the axis of the cone the line of intersection of the plane and the surface of the cone is a circle. Refer to Figure 133a for a depiction of conic sections.

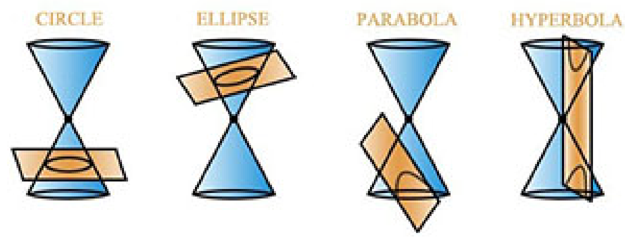

Figure 133a. Conic sections.

If an intersecting plane is tilted to some position, the intersection is an **ellipse** or flattened circle, see Figure 133b. The longest diameter of an ellipse is called its **major axis**, and half of this is its **semimajor axis**, which is identified by the letter "**a**" in Figure 133b. The shortest diameter of an ellipse is called its **minor axis**, and half of this is its semiminor axis, which is identified by the letter "**b**" in figure Figure 133b. Two points, F and F', called **foci** (singular **focus**) or **focal points**, on the major axis are so located that the sum of their distances from any point P on the curve is equal to the length of the major axis. That is $PF + PF' = 2a$ (Figure 133b). The **eccentricity** (**e**) of an ellipse is equal to $\frac{c}{a}$, where c is the distance from the center to one of the foci ($c = CF = CF'$). It is always greater than 0 but less than 1.

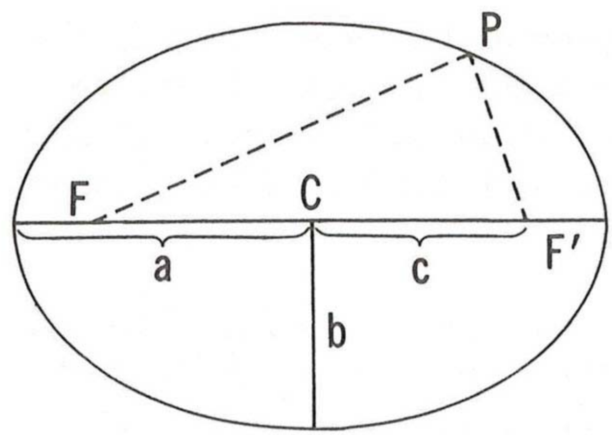

Figure 133b. An ellipse.

If an intersecting plane is parallel to one element of the cone the intersection is a **parabola**, See Figure 133c Any point P on a parabola is equidistant from a fixed point F, called the **focus** or **focal point**, and a fixed straight line, AB, called the **directrix**. Thus, for any point P, $PF = PE$. The point midway between the focus F and the directrix AB is called the **vertex**, V. The straight line through F and V is called the **axis**, CD. This line is perpendicular to the directrix AB. The **eccentricity** (e) of a parabola is 1.

If the elements of the cone are extended to form a second cone having the same axis and apex but extending in the opposite direction, and the intersecting plane is tilted

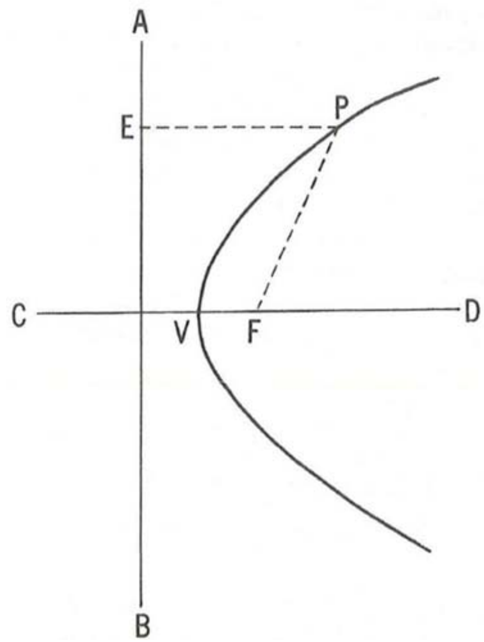

Figure 133c. A parabola.

beyond the position forming a parabola, so that it intersects both curves, the intersections of the plane with the cones is a **hyperbola**, See Figure 133d. There are two intersections or branches of a hyperbola, as shown. At any point P on either branch, the difference in the distance from two fixed points called **foci** or **focal points**, F and F', is constant and equal to the shortest distance between the two branches. That is, $PF - PF' = 2a$ (Figure 133d). The straight line through F and F' is called the **axis**. The **eccentricity** (**e**) of a hyperbola is the ratio $\frac{c}{a}$ (Figure 133d). It is always greater than 1.

Each branch of a hyperbola approaches ever closer to, but never reaches, a pair of intersecting straight lines, AB and CD, called **asymptotes**. These intersect at G.

The various conic sections bear an eccentricity relationship to each other. The eccentricity of a circle is 0, that of an ellipse is greater than 0 but less than 1; that of a parabola or straight line (a limiting case of a parabola) is 1, and that of a hyperbola is greater than 1.

If e = eccentricity, A = area, a = semimajor axis of an ellipse or half the shortest distance between the two branches of a hyperbola, b = the semiminor axis of an ellipse, and c = the distance between the center of an ellipse and one of its focal points or the distance between the focal point of a hyperbola and the intersection of its asymptotes:

Circle:
Eccentricity: $e = 0$
Ellipse:
Area: $A = \pi ab$
Eccentricity: $e = \frac{c}{a}$, greater that 0, but less than 1.
Parabola:
Eccentricity: $e = 1$
Hyperbola:
Eccentricity: $e = \frac{c}{a}$, greater than 1.

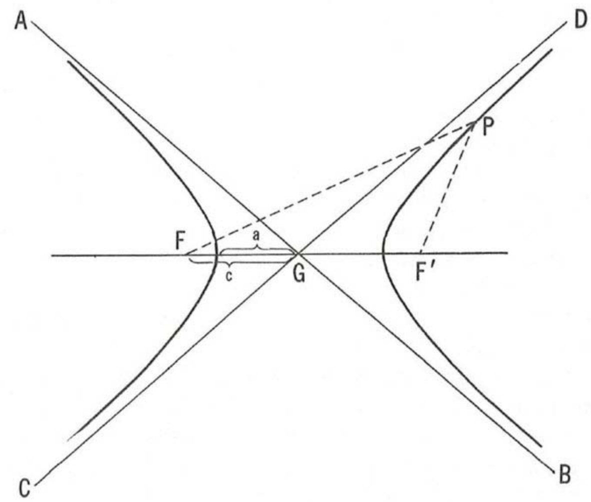

Figure 133d. A hyperbola.

When cones are intersected by some surface other than a plane, as the curved surface of the earth, the resulting sections do not follow the relationships given above, the amount of divergence therefrom depending upon the individual circumstances.

134. Spheres

A **sphere** is a solid bounded by a surface every point of which is equidistant from a point within called the **center**. It may also be formed by rotating a circle about any diameter.

A **radius** or **semidiameter** of a sphere is a straight line connecting its center with any point on its surface. A **diameter** of a sphere is a straight line through its center and terminated at both ends by the surface of the sphere. The poles of a sphere are the ends of a diameter.

The intersection of a plane and the surface of a sphere is a circle, a **great circle** if the plane passes through the center of the sphere, and a **small circle** if it does not. The shorter arc of the great circle between two points on the surface of a sphere is the shortest distance, on the surface of the sphere, between the points. Every great circle of a sphere bisects every other great circle of that sphere. The **poles** of a circle on a sphere are the extremities of the sphere's diameter which is perpendicular to the plane of the circle. All points on the circumference of the circle are equidistant from either of its poles. In the ease of a great circle, *both* poles are 90° from any point on the circumference of the circle. Any great circle may be considered a **primary**, particularly when it serves as the origin of measurement of a coordinate. The great circles through its poles are called **secondary**. Secondaries are perpendicular to their primary.

A **spherical triangle** is the figure formed on the surface of a sphere by the intersection of three great circles. The lengths of the sides of a spherical triangle are measured in degrees, minutes, and seconds, as the angular lengths of the arcs forming them. The sum of the three sides is always less than 360°. The sum of the three angles is always *more* than 180° and *less* than 540°.

A **lune** is the part of the surface of a sphere bounded by halves of two great circles.

A **spheroid** is a flattened sphere, which may be formed by rotating an ellipse about one of its axes. An **oblate spheroid**, such as the earth, is formed when an ellipse is rotated about its minor axis. In this case the diameter along the axis of rotation is less than the major axis. A **prolate spheroid** is formed when an ellipse is rotated about its major axis. In this case the diameter along the axis of rotation is greater than the minor axis.

If A = area, r = radius, d = diameter, and V = volume of a sphere:

Area: $A = 4\pi r^2 = \pi d^2$

Volume: $V = \frac{4\pi r^3}{3} = \frac{\pi d^3}{6}$

If A = area, a = semimajor axis, b = semiminor axis, e = eccentricity, and V = volume of an oblate spheroid:

Area: $A = 4\pi a^2\left(1 - \frac{e^2}{3} - \frac{e^4}{15} - \frac{e^6}{35} - \dots\right)$

Eccentricity: $e = \sqrt{\frac{a^2 - b^2}{a^2}}$

Volume: $V = \frac{4\pi a^2 b}{3}$

135. Coordinates

Coordinates are magnitudes used to define a position. Many different types of coordinates are used. Important navigational ones are described below.

If a position is known to be at a stated point, no magnitudes are needed to identify the position, although they may be required to locate the point. Thus, if a vessel is at port A, its position is known if the location of port A is known, but latitude and, longitude may be needed to locate port A.

If a position is known to be on a given line, a single magnitude (coordinate) is needed to identify the position if an origin is stated or understood. Thus, if a vessel is known to be *south* of port B, it is known to be on a line extending southward from port B. If its distance from port B is known, and the position of port B is known, the position of the vessel is uniquely defined.

If a position is known to be on a given surface, two magnitudes (coordinates) are needed to define the position. Thus, if a vessel is known to be on the surface of the earth,

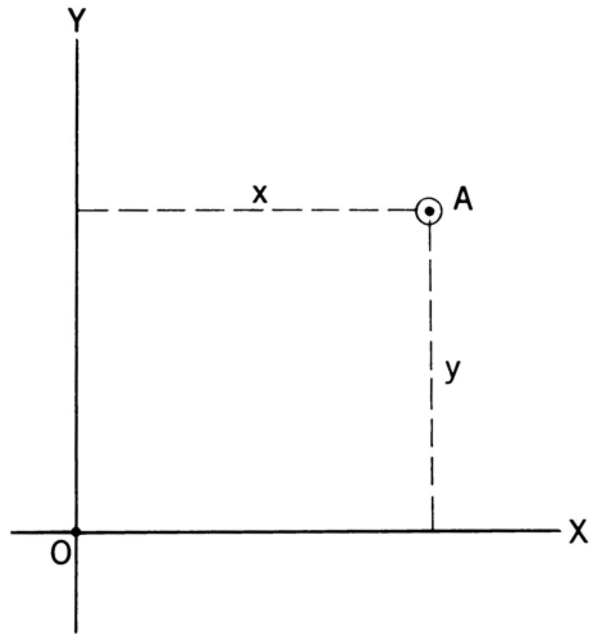

Figure 135a. Rectangular coordinates.

its position can be identified by means of latitude and longitude. Latitude indicates its angular distance north or south of the equator, and longitude its angular distance east or west of the prime meridian.

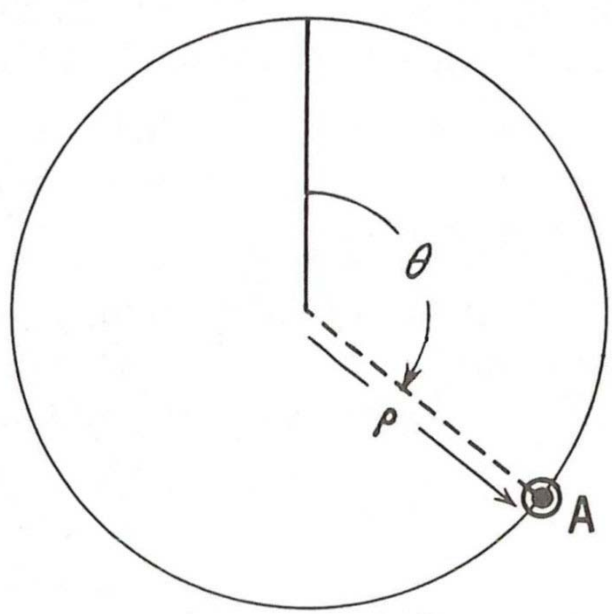

Figure 135b. Polar coordinates.

If nothing is known regarding a position other than that it exists in space, three magnitudes (coordinates) are needed to define its position. Thus, the position of a submarine may be defined by means of latitude, longitude, and depth below the surface.

Each coordinate requires an origin, either stated or implied. If a position is known to be on a given plane, it might be defined by means of its distance from each of two intersecting lines, called **axes**. These are called **rectangular coordinates**. In Figure 135a, *OY* is called the **ordinate**, and *OX* is called the **abscissa**. Point *O* is the **origin**, and lines *OX* and *OY* the axes (called the *X* and *Y* axes, respectively). Point *A* is at position *x*, *y*. If the axes are not perpendicular but the lines *x* and *y* are drawn parallel to the axes, **oblique coordinates** result. Either type are called **Cartesian coordinates**. A three–dimensional system of Cartesian coordinates, with *X*, *Y*, and *Z* axes, is called **space coordinates**.

Another system of plane coordinates in common usage consists of the *direction* and *distance* from the origin (called the **pole**), as shown in Figure 135b. A line extending in the direction indicated is called a **radius vector**. Direction and distance from a fixed point constitute **polar coordinates**, sometimes called the rho– (the Greek ρ, to indicate distance) theta (the Greek θ, to indicate direction) system. An example of its use is the radar scope.

Spherical coordinates are used to define a position on the surface of a sphere or spheroid by indicating angular distance from a primary great circle and a reference secondary great circle. Examples used in navigation are latitude and longitude, altitude and azimuth, and declination and hour angle.

TRIGONOMETRY

136. Definitions

Trigonometry deals with the relations among the angles and sides of triangles. **Plane trigonometry** deals with plane triangles, those on a plane surface. Spherical trigonometry deals with spherical triangles, which are drawn on the surface of a sphere. In navigation, the common methods of celestial sight reduction use spherical triangles on the surface of the Earth. For most navigational purposes, the Earth is assumed to be a sphere, though it is somewhat flattened.

137. Angular Measure

A circle may be divided into 360 **degrees** (°), which is the **angular length** of its circumference. Each degree may be divided into 60 **minutes** ('), and each minute into 60 **seconds** ("). The angular measure of an arc is usually expressed in these units. By this system a right angle or quadrant has 90° and a straight angle or semicircle 180°. In marine navigation, altitudes, latitudes, and longitudes are usually expressed in degrees, minutes, and tenths (27°14.4'). Azimuths are usually expressed in degrees and tenths (164.7°). The system of degrees, minutes, and seconds indicated above is the **sexagesimal system**. In the **centesimal system,** used chiefly in France, the circle is divided into 400 **centesimal degrees** (sometimes called **grades**) each of which is divided into 100 centesimal minutes of 100 **centesimal seconds** each.

A **radian** is the angle subtended at the center of a circle by an arc having a linear length equal to the radius of the circle. A radian is equal to 57.2957795131° approximately, or 57°17'44.80625" approximately. The radian is sometimes used as a unit of angular measure. See Figure 137. A circle $(360°) = 2\pi$ radians, a semicircle $(180°) = \pi$ radians, a right angle measure $(90°) = \frac{\pi}{2}$ radians, and 1' = 0.0002908882 radians approximately.The length of the arc of a circle is equal to the radius multiplied by the angle subtended in radians.

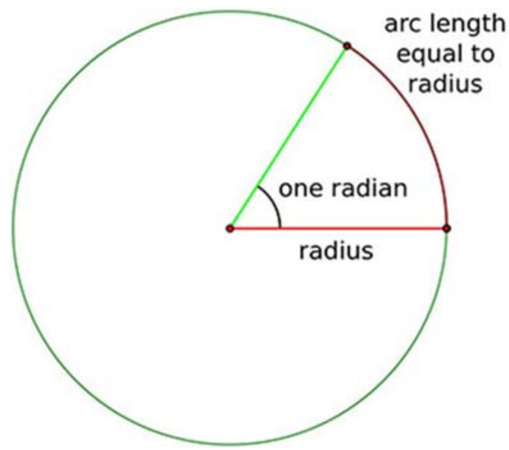

Figure 137. Image depicting one radian.

138. Trigonometry

Trigonometry is that branch of mathematics dealing with the relations among the angles and sides of triangles. **Plane trigonometry** is that branch dealing with plane triangles, and **spherical trigonometry** is that branch dealing with spherical triangles.

Trigonometric functions are the various proportions or ratios of the sides of a plane right triangle, defined in relation to one of the acute angles. In Figure 138a, let θ be any acute angle. From any point R on line OA, draw a line perpendicular to OB at F. From any other point R' on OA, draw a line perpendicular to OB at F'. Then triangles OFR and OF'R' are similar right triangles because all their corresponding *angles* are equal. Since in any pair of similar triangles the ratio of any two sides of one triangle is equal to

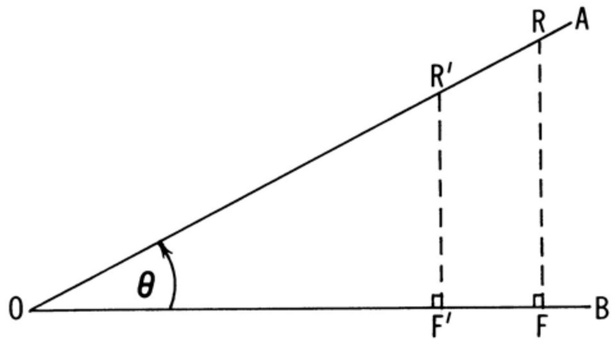

Figure 138a. Similar right triangles.

the ratio of the corresponding two sides of the other triangle,

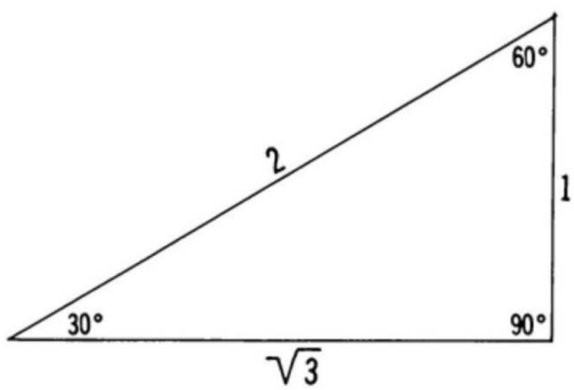

Figure 138b. *Numerical relationship of sides of a 30°-60°-90° triangle.*

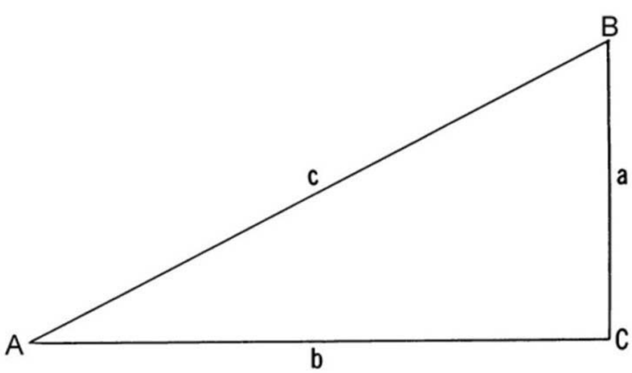

Figure 138c. A right triangle.

$$\frac{RF}{OF} = \frac{R'F'}{OF'},\ \frac{RF}{OR} = \frac{R'F'}{OR'},\ \text{and}\ \frac{OF}{OR} = \frac{OF'}{OR'}$$

No matter where the point R is located on OA, the ratio between the lengths of any two sides in the triangle OFR has a constant value. Hence, for any value of the acute angle θ, there is a fixed set of values for the ratios of the various sides of the triangle. These ratios are defined as follows:

$$\text{sine } \theta = \sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}$$

$$\text{cosine } \theta = \cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}$$

$$\text{tangent } \theta = \tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}$$

$$\text{cosecant } \theta = \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite}}$$

$$\text{secant } \theta = \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}}$$

$$\text{cotangent } \theta = \cot\theta = \frac{\text{side adjacent}}{\text{side opposite}}$$

Of these six principal functions, the second three are the reciprocals of the first three; therefore

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

In Figure 138c, *A*, *B*, and *C* are the angles of a plane right triangle, with the right angle at *C*. The sides are labeled *a*, *b*, *c*, with opposite angles labeled *A*, *B*, and *C* respectively.

The six principal trigonometric functions of angle *B* are:

$$\sin B = \frac{b}{c} = \cos A = \cos(90° - B)$$

$$\cos B = \frac{a}{c} = \sin A = \sin(90° - B)$$

$$\tan B = \frac{b}{a} = \cot A = \cot(90° - B)$$

$$\cot B = \frac{a}{b} = \tan A = \tan(90° - B)$$

$$\sec B = \frac{c}{a} = \csc A = \csc(90° - B)$$

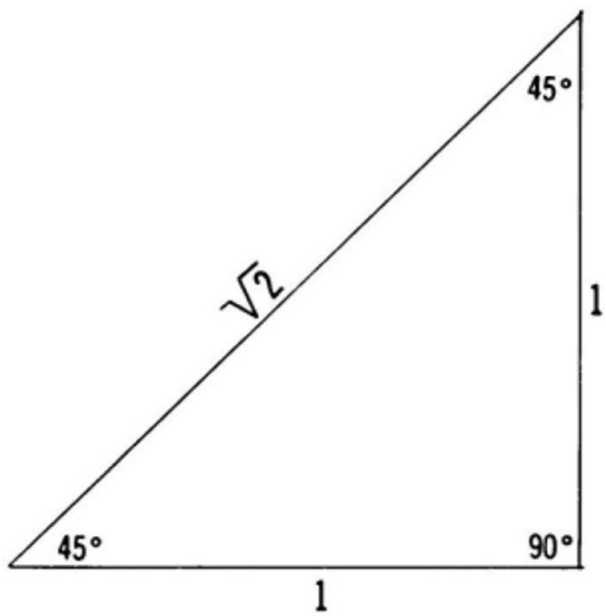

Figure 138d. Numerical relationship of sides of a 45°-45°-90° triangle.

$$\csc B = \frac{c}{b} = \sec A = \sec(90° - B)$$

Function	*30°*	*45°*	*60°*
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$	$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$
cosine	$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}$	$\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
tangent	$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$	$\frac{1}{1} = 1$	$\frac{\sqrt{3}}{1} = \sqrt{3}$
cotangent	$\frac{\sqrt{3}}{1} = \sqrt{3}$	$\frac{1}{1} = 1$	$\frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$

Table 138e. Values of various trigonometric functions for angles 30°, 45°, and 60°.

Function	*30°*	*45°*	*60°*
secant	$\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{1} = 2$
cosecant	$\frac{2}{1} = 2$	$\frac{\sqrt{2}}{1} = \sqrt{2}$	$\frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$

Table 138e. Values of various trigonometric functions for angles 30°, 45°, and 60°.

Since *A* and *B* are *complementary*, these relations show that the sine of an angle is the cosine of its complement, the tangent of an angle is the cotangent of its complement, and the secant of an angle is the cosecant of its complement. Thus, the **co**-function of an angle is the function of its complement.

$$\sin(90° - A) = \cos A$$

$$\cos(90° - A) = \sin A$$

$$\tan(90° - A) = \cot A$$

$$\cot(90° - A) = \tan A$$

$$\sec(90° - A) = \csc A$$

$$\csc(90° - A) = \sec A$$

Certain additional relations are also classed as trigonometric functions:

versed sine θ = versine θ = vers θ = ver θ = 1- cos θ

versed cosine θ = coversed sine θ

(therefore) coversed sine θ = coversine θ

(therefore) coversine θ = covers θ

(therefore) covers θ = cov θ

(therefore) cov θ =1 - sin θ

haversine θ = hav θ = $1/2$ ver θ = $(1/2)(1 - \cos\theta)$.

The numerical value of a trigonometric function is sometimes called the **natural function** to distinguish it from the logarithm of the function, called the **logarithmic function**. Numerical values of the six principal functions are given at 1' intervals in Table 2- Natural Trigonometric Functions. Logarithms are given at the same intervals in Table 3- Common Logarithms of Trigonometric Functions.

Since the relationships of 30°, 60°, and 45° right triangles are as shown in Figure 138c and Figure 138b, certain values of the basic functions can be stated exactly, as shown in Table 138e.

All trigonometric functions can be shown as lengths of lines in a unit circle. See Figure 138f for a depiction of the following equations:

$$\sin\theta = RF$$
$$\cot\theta = AB$$
$$\cos\theta = OF$$
$$\sec\theta = OD$$
$$\tan\theta = DE$$
$$\csc\theta = OA$$
$$\text{ver}\,\theta = FE$$
$$\text{cov}\,\theta = BC$$

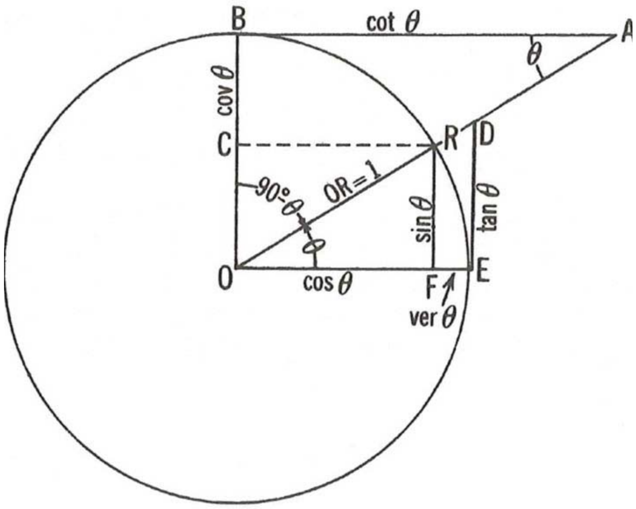

Figure 138f. Line definitions of trigonometric functions.

139. Functions in Various Quadrants

To make the definitions of the trigonometric functions more general to include those angles greater than 90°, the functions are defined in terms of the rectangular Cartesian coordinates of point R of Figure 138a, due regard being given to the sign of the function. In Figure 139a, OR is assumed to be a *unit* radius. By convention the sign of OR is always positive. This radius is imagined to rotate in a counterclockwise direction through 360° from the horizontal position at 0°, the positive direction along the X axis. Ninety degrees (90°) is the positive direction along the Y axis. The angle between the original position of the radius and its position at any time increases from 0° to 90° in the *first quadrant* (I), 90° to 180° in the *second quadrant* (II), 180° to 270° in the *third quadrant* (III), and 270° to 360° in the *fourth quadrant* (IV).

The numerical value of the sine of an angle is equal to the projection of the unit radius on the Y–axis. According to the definition given in Section138, the sine of angle in the first quadrant of Figure 139a is $\frac{+y}{+OR}$. If the radius OR is equal to one, sin θ =+y. Since +y is equal to the projection of the unit radius OR on the Y axis, the sine function of an angle in the first quadrant defined in terms of rectangular Cartesian coordinates does not contradict the definition in Section 138. In Figure 139a,

$$\sin\theta = +y$$

$$\sin(180°-\theta) = +y = \sin\theta$$

$$\sin(180°+\theta) = -y = -\sin\theta$$

$$\sin(360°-\theta) = -y = \sin(-\theta) = -\sin\theta$$

The numerical value of the cosine of an angle is equal to the projection of the unit radius on the X axis. In Figure 139a,

$$\cos\theta = +x$$
$$\cos(180°-\theta) = -x = -\cos\theta$$
$$\cos(180°+\theta) = -x = -\cos\theta$$
$$\cos(360°-\theta) = +x = \cos(-\theta) = \cos\theta$$

The numerical value of the tangent of an angle is equal to the ratio of the projections of the unit radius on the Y and X axes. In Figure 139a,

$$\tan\theta = \frac{+y}{+x}$$

$$(180°-\theta) = \frac{+y}{-x} = -\tan\theta$$

$$\tan(180°+\theta) = \frac{-y}{-x} = \tan\theta$$

$$\tan(360°-\theta) = \frac{-y}{+x} = \tan(-\theta) = -\tan\theta$$

The cosecant, secant, and cotangent functions of angles in the various quadrants are similarly determined:

$$\csc\theta = \frac{1}{+y}$$

$$\csc(180° - \theta) = \frac{1}{+y} = \csc\theta$$

$$\csc(180° + \theta) = \frac{1}{-y} = -\csc\theta$$

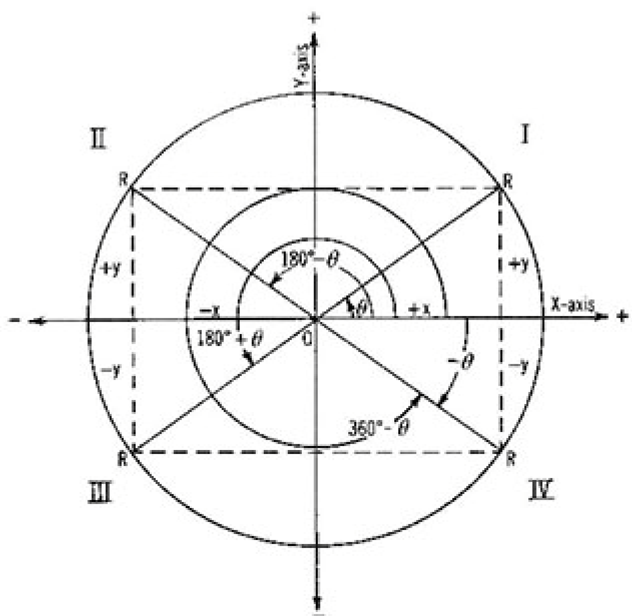

Figure 139a. The functions in various quadrants, mathematical convention.

$$\csc(360°-\theta) = \frac{1}{-y} = \csc(-\theta) = -\csc\theta$$

$$\sec\theta = \frac{1}{+x}$$

$$\sec(180°-\theta) = \frac{1}{-x} = -\sec\theta$$

$$\sec(180°+\theta) = \frac{1}{-x} = -\sec\theta$$

$$\sec(360°-\theta) = \frac{1}{+x} = \sec(-\theta) = \sec\theta$$

$$\cot\theta = \frac{+x}{+y}$$

$$\cot(180°-\theta) = \frac{-x}{+y} = -\cot\theta$$

$$\cot(180°+\theta) = \frac{-x}{-y} = \cot\theta$$

$$\cot(360°-\theta) = \frac{+x}{-y} = \cot(-\theta) = -\cot\theta.$$

The signs of the functions in the four different quadrants are shown below:

	I	II	III	IV
sine and cosecant	+	+	-	-
cosine and secant	+	-	-	+
tangent and cotangent	+	-	+	-

Table 139b. Signs of trigonometric functions by quadrants.

These relationships are shown in Table 139c and graphically in Figure 139d through Figure 139f.

	I	II	III	IV
sin	0 to +1	+1 to 0	0 to −1	−1 to 0
csc	+∞ to +1	+1 to 0	−∞ to −1	−1 to −∞
cos	+1 to 0	0 to −1	−1 to 0	0 to +1
sec	+1 to +∞	−∞ to −1	−1 to −∞	+∞ to +1
tan	0 to +∞	−∞ to 0	0 to +∞	−∞ to 0
cot	+∞ to 0	0 to −∞	+∞ to 0	0 to −∞

Table 139c. Values of trigonometric functions in various quadrants.

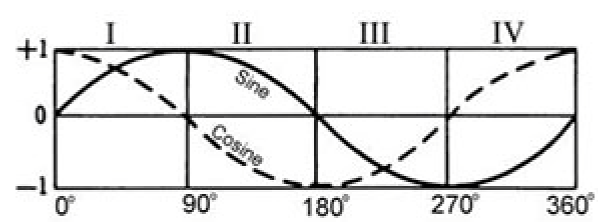

Figure 139d. Graphic representation of values of trigonometric functions in various quadrants.

The numerical values vary by quadrant as shown above:

As shown in Figure 139a and Table 139b, the sign (+ or -) of the functions varies with the quadrant of an angle. In Figure 139a radius OR is imagined to rotate in a counter-

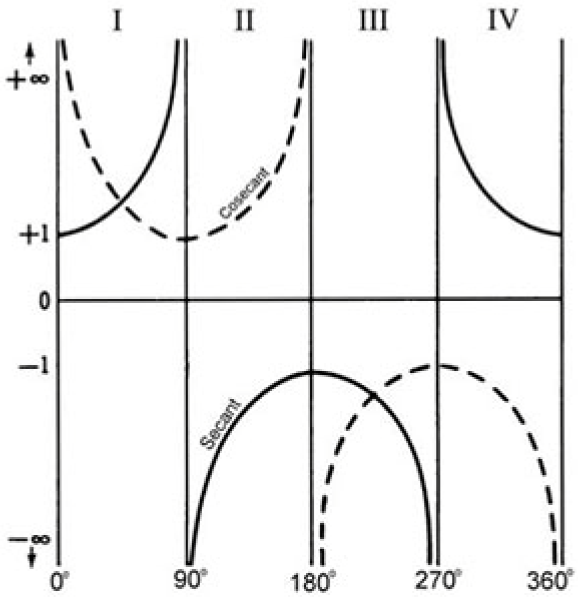

Figure 139e. Graphic representation of values of trigonometric functions in various quadrants.

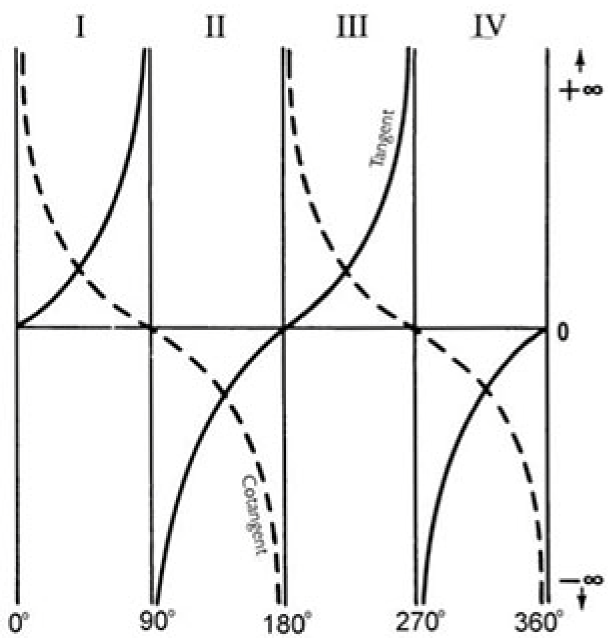

Figure 139f. Graphic representation of values of trigonometric functions in various quadrants.

clockwise direction through 360° from the horizontal position at 0°. This is the **mathematical convention**. In Figure 139h this concept is shown in the usual **navigational convention** of a compass rose, starting with 000° at the top and rotating clockwise. In either diagram the angle θ be-

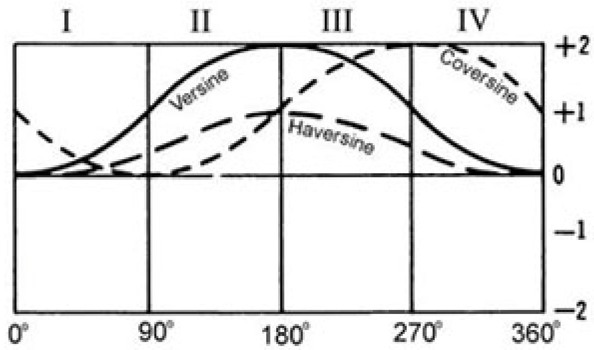

Figure 139g. Graphic representation of values of trigonometric functions in various quadrants.

tween the original position of the radius and its position at any time increases from 0° to 90° in the *first quadrant* (I), 90° to 180° in the *second quadrant* (II), 180° to 270° in the *third quadrant* (III), and 270° to 360° in the *fourth quadrant* (IV). Also in either diagram, 0° is the positive direction along the X-axis. Ninety degrees (90°) is the positive direction along the Y-axis. Therefore, the projections of the unit radius OR on the X- and Y-axes, as appropriate, produce the same values of the trigonometric functions.

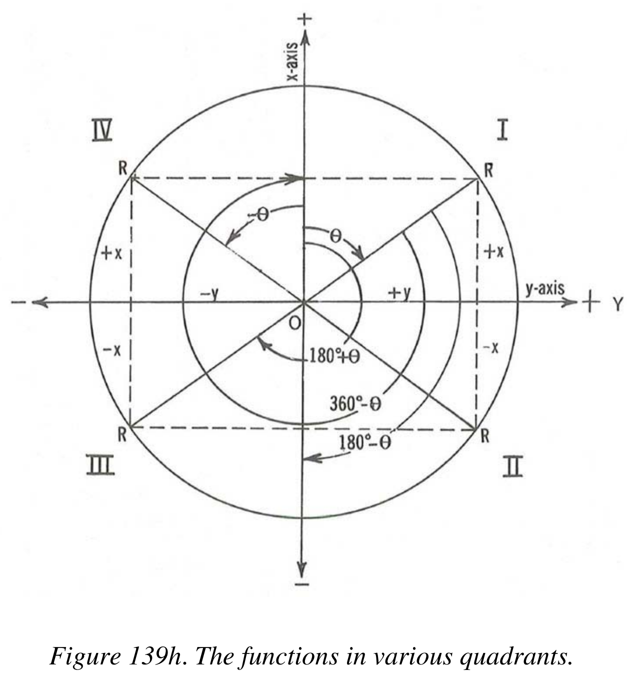

Figure 139h. The functions in various quadrants.

A **negative angle (-θ)** is an angle measured in a clockwise direction (mathematical convention) or in a direction opposite to that of a positive angle. The functions of a negative angle and the corresponding functions of a positive angle are as follows:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan-\theta = -\tan\theta$

$\tan(-\theta) = \tan(360° - \theta)$

140. Trigonometric Identities

A **trigonometric identity** is an equality involving trigonometric functions of θ which is true for all values of θ, except those values for which one of the functions is not defined or for which a denominator in the equality is equal to zero. The **fundamental identities** are those identities from which other identities can be derived.

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

141. Reduction Formulas

$$\sin(90° - \theta) = \cos\theta \qquad \csc(90° - \theta) = \sec\theta$$
$$\cos(90° - \theta) = \sin\theta \qquad \sec(90° - \theta) = \csc\theta$$
$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \csc(-\theta) = -\csc\theta$$
$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$
$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\sin(90+\theta) = \cos\theta \qquad \csc(90+\theta) = \sec\theta$$
$$\cos(90+\theta) = -\sin\theta \qquad \sec(90+\theta) = -\csc\theta$$
$$\tan(90+\theta) = -\cot\theta \qquad \cot(90+\theta) = -\tan\theta$$

$$\sin(180°+\theta) = -\sin\theta \qquad \csc(180°+\theta) = -\csc\theta$$
$$\cos(180°+\theta) = -\cos\theta \qquad \sec(180°+\theta) = -\sec\theta$$
$$\tan(180°+\theta) = \tan\theta \qquad \cot(180°+\theta) = \cot\theta$$

$$\sin(360° - \theta) = -\sin\theta \qquad \csc(360° - \theta) = -\csc\theta$$
$$\cos(360° - \theta) = \cos\theta \qquad \sec(360° - \theta) = \sec\theta$$
$$\tan(360° - \theta) = -\tan\theta \qquad \cot(360° - \theta) = -\cot\theta$$

142. . Inverse Trigonometric Functions

An angle having a given trigonometric function may be indicated in any of several ways. Thus, sin $y = x$, y = arc sin x, and $y = \sin^{-1} x$ have the same meaning. The superior "–1" is not an exponent in this case. In each case, y is "the angle whose sine is x." In this case, y is the **inverse sine** of x. Similar relationships hold for all trigonometric functions.

SOLVING TRIANGLES

Solution of triangles. A triangle is composed of six parts: three angles and three sides. The angles may be designated *A*, *B*, and *C*; and the sides opposite these angles as *a*, *b*, and *c*, respectively. In general, when any three parts are known, the other three parts can be found, unless the known parts are the three angles of a plane triangle.

143. Right Plane Triangles

In a **right plane triangle** it is only necessary to substitute numerical values in the appropriate formulas representing the basic trigonometric functions and solve. Thus, if a and b are known,

$$\tan A = \frac{a}{b}$$
$$B = 90° - A$$
$$c = a \csc A$$

Similarly, if c and B are given,

$$A = 90° - B$$
$$a = c \sin A$$
$$b = c \cos A$$

144. Oblique Plane Triangles

When solving an **oblique plane triangle**, it is often desirable to draw a rough sketch of the triangle approximately to scale, as shown in Figure 144. The following laws are helpful in solving such triangles:

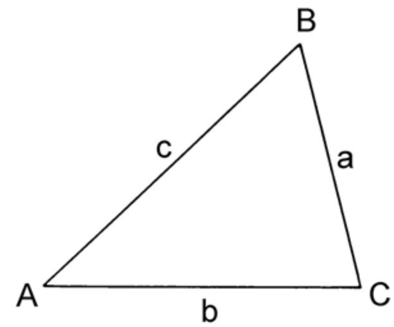

Figure 144. An oblique plane triangle.

Known	*To find*	*Formula*	*Comments*
a, b, c	*A*	$\cos A = \frac{c^2+b^2-a^2}{2bc}$	Cosine law
a, b, A	*B*	$\sin B = \frac{b \sin A}{a}$	Sine law. Two solutions if *b>a*
	C	$C = 180° - (A+B)$	$A + B + C = 180°$
	c	$c = \frac{a \sin C}{\sin A}$	Sine law
a, b, C	*A*	$\tan A = \frac{a \sin C}{b - a \cos C}$	
	B	$B = 180° - (A+C)$	$A + B + C = 180°$
	c	$c = \frac{a \sin C}{\sin A}$	Sine law
a, A, B	*b*	$b = \frac{a \sin B}{\sin A}$	Sine law
	C	$C = 180° - (A+B)$	$A + B + C = 180°$
	c	$c = \frac{a \sin C}{\sin A}$	Sine law

Table 144. Formulas for solving oblique plane triangles.

Law of sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc \cos A.$

The unknown parts of oblique plane triangles can be computed by the formulas in Table 144, among others. By reassignment of letters to sides and angles, these formulas can be used to solve for all unknown parts of oblique plane triangles.

SPHERICAL TRIGONOMETRY

145. Napier's Rules

Right spherical triangles can be solved with the aid of **Napier's Rules of Circular Parts**. If the right angle is omitted, the triangle has five parts: two angles and three sides, as shown in Figure 145a. Since the right angle is already known, the triangle can be solved if any two other parts are known. If the two sides forming the right angle, and the *complements* of the other three parts are used, these elements (called "parts" in the rules) can be arranged in five sectors of a circle in the same order in which they occur in the triangle, as shown in Figure 145b. Considering *any* part as the *middle* part, the two parts nearest it in the diagram are considered the *adjacent* parts, and the two farthest from it the *opposite* parts.

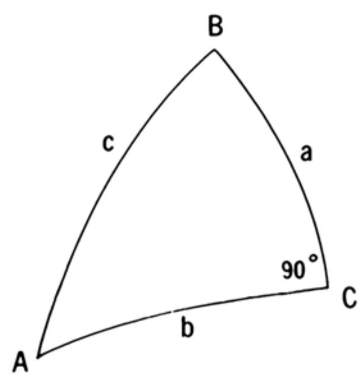

Figure 145a. Parts of a right spherical triangle as used in Napier's rules.

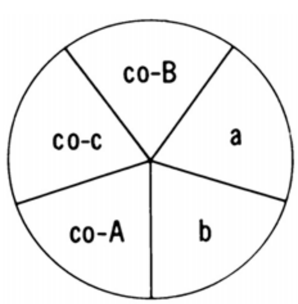

Figure 145b. Diagram for Napier's Rules of Circular Parts.

The following rules apply:

Napier's Rules state: The sine of a middle part equals the product of (1) the tangents of the adjacent parts or (2) the cosines of the opposite parts.

In the use of these rules, the co-function of a complement can be given as the function of the element. Thus, the cosine of co–A is the same as the sine of A. From these rules the following formulas can be derived:

$$\sin a = \tan b \cot B = \sin c \sin A$$
$$\sin b = \tan a \cot A = \sin c \sin B$$
$$\cos c = \cot A \cot B = \cos a \cos b$$
$$\cos A = \tan b \cot c = \cos a \sin B$$
$$\cos B = \tan a \cot c = \cos b \sin A$$

1. An oblique angle and the side opposite are in the same quadrant.
2. Side c (the hypotenuse) is less then 90° when a and b are in the same quadrant, and more than 90° when a and b are in different quadrants.

If the known parts are an angle and its opposite side, two solutions are possible.

A **quadrantal spherical triangle** is one having one side of 90°. A **biquadrantal spherical triangle** has two sides of 90°. A **triquadrantal spherical triangle** has three sides of 90°. A biquadrantal spherical triangle is isosceles and has two right angles opposite the 90° sides. A triquadrantal spherical triangle is equilateral, has three right angles, and bounds an octant (one–eighth) of the surface of the sphere. A quadrantal spherical triangle can be solved by Napier's rules provided any two elements in addition to the 90° side are known. The 90° side is omitted and the other parts are arranged in order in a five–sectored circle, using the complements of the three parts farthest from the 90° side. In the case of a quadrantal triangle, rule 1 above is used, and rule 2 restated: angle C (the angle opposite the side of 90°) is *more* than 90° when A and B are in the same quadrant, and *less* than 90° when A and B are in different quadrants. If the rule requires an angle of more than 90° and the solution produces an angle of less than 90°, subtract the solved angle from 180°.

146. Oblique Spherical Triangles

An **oblique spherical triangle** can be solved by dropping a perpendicular from one of the apexes to the opposite side, subtended if necessary, to form two right spherical triangles. It can also be solved by the following formulas in Table 146, reassigning the letters as necessary.

Known	*To find*	*Formula*	*Comments*
a, b, C	A	$\tan \mathrm{A} = \dfrac{\sin D \tan \mathrm{C}}{\sin (b - D)}$	$\tan D = \tan a \cos \mathrm{C}$
	B	$\sin \mathrm{B} = \dfrac{\sin \mathrm{C} \sin b}{\sin c}$	
$c, \mathrm{A}, \mathrm{B}$	C	$\cos \mathrm{C} = \sin \mathrm{A} \sin \mathrm{B} \cos c - \cos \mathrm{A} \cos \mathrm{B}$	
	a	$\tan a = \dfrac{\tan c \sin E}{\sin (\mathrm{B} + E)}$	$\tan E = \tan \mathrm{A} \cos c$
	b	$\tan b = \dfrac{\tan c \sin F}{\sin (\mathrm{A} + F)}$	$\tan F = \tan \mathrm{B} \cos c$
a, b, A	c	$\sin (c + G) = \dfrac{\cos a \sin G}{\cos b}$	$\cot G = \cos \mathrm{A} \tan b$ Two solutions
	B	$\sin \mathrm{B} = \dfrac{\sin \mathrm{A} \sin b}{\sin \mathrm{a}}$	Two solutions
	C	$\sin (\mathrm{C} + H) = \sin H \tan b \cot a$	$\tan H = \tan \mathrm{A} \cos b$ Two solutions
$a, \mathrm{A}, \mathrm{B}$	C	$\sin (\mathrm{C} - K) = \dfrac{\cos \mathrm{A} \sin K}{\cos \mathrm{B}}$	$\cot K = \tan \mathrm{B} \cos a$ Two solutions
	b	$\sin b = \dfrac{\sin a \sin \mathrm{B}}{\sin \mathrm{A}}$	Two solutions
	c	$\sin (c - M) = \cot \mathrm{A} \tan \mathrm{B} \sin M$	$\tan M = \cos \mathrm{B} \tan a$ Two solutions

Table 146. Formulas for solving oblique spherical triangles.

147. . Other Useful Formulas

In addition to the fundamental trigonometric identities and reduction formulas given in Section 139, the following formulas apply to plane and spherical trigonometry:

Addition and Subtraction Formulas

$$\sin (\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$$

$$\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

$$\sin (\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi$$

$$\cos (\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$$

$$\tan (\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} .$$

Double-Angle Formulas

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} .$$

Half-Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

The following are useful formulas of spherical trigonometry:

Law of Cosines for Sides

cos a = cos b cos c + sin b sin c cos A

cos b = cos c cos a + sin c sin a cos B

cos c = cos a cos b + sin a sin b cos C

Law of Cosines for Angles

cos A= - cos B cos C + sin B sin C cos a

cos B = - cos C cos A + sin C sin A cos b

cos C= - cos A cos B + sin A sin B cos c.

Law of Sines

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Napier's Analogies

$$\tan\frac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{1}{2}C$$

$$\tan\frac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{1}{2}C$$

$$\tan\frac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{1}{2}c$$

$$\tan\frac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{1}{2}c.$$

Five Parts Formulas

sin a cos B = cos b sin c - sin b cos c cos A

sin b cos C= cos c sin a - sin c cos a cos B

sin c cos A= cos a sin b - sin a cos b cos C.

Haversine Formulas

hav a = hav ($b \sim c$) + sin b sin c hav A

hav b = hav ($a \sim c$) + sin a sin c hav B

hav c = hav ($a \sim b$) + sin a sin b hav C

hav A= [hav a- hav ($b \sim c$)] csc b csc c

hav B = [hav b- hav ($a \sim c$)] csc a csc c

hav C = [hav c- hav ($a \sim b$)] csc a csc b.

148. Functions of a Small Angle

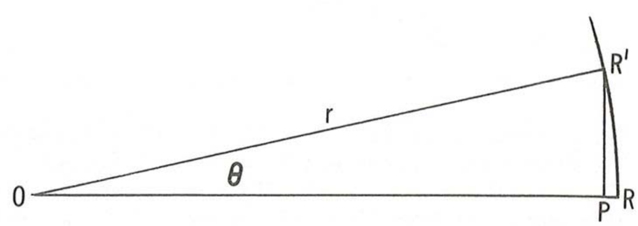

Figure 148. A small angle.

Functions of a small angle: In Figure 148, small angle θ, measured in radians, is subtended by the arc RR' of a circle. The radius of the circle is r, and $R'P$ is perpendicular to OR at P. Since the length of the arc of a circle is equal to the radius multiplied by the angle subtended in radians,

$RR' = r \times \theta$.

When θ is sufficiently small for $R'P$ to approximate RR',

$$\sin\theta = \theta$$

since $\theta = \dfrac{RR'}{r}$ and $\sin\theta = \dfrac{R'P}{r}$.

For small angles, it can also be shown that

$$\tan\theta = \theta.$$

If there are x minutes of arc (x') in a small angle of θ radians,

$$\sin x' = x \sin 1'.$$

Figure 148 also shows that when θ is small, *OP* is approximately equal to the radius. Therefore, cos θ can be taken as equal to 1.

Another approximation can be obtained if cos θ is expressed in terms of the half-angle:

$$\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$$

$$\cos\theta = 1 - 2\left(\frac{1}{2}\theta\right)^2$$

$$\cos\theta = 1 - \frac{1}{2}\theta^2.$$

CALCULUS

149. Calculus

Calculus is that branch of mathematics dealing with the rate of change of one quantity with respect to another.

A **constant** is a quantity which does not change. If a vessel is making good a course of 090°, the latitude does not change and is therefore a constant.

A **variable**, where continuous, is a quantity which can have an infinite number of values, although there may be limits to the maximum and minimum. Thus, from latitude 30° to latitude 31 ° there are an infinite number of latitudes, if infinitesimally small units are taken, but no value is less than 30° nor more than 31°. If two variables are so related that for every value of one there is a corresponding value of the other, one of the values is known as a **function** of the other. Thus, if speed is constant, the distance a vessel steams depends upon the elapsed time. Since elapsed time does not depend upon any other quantity, it is called an **independent variable**. The distance depends upon the elapsed time, and therefore is called a **dependent variable**. If it is required to find the time needed to travel any given distance at constant speed, distance is the independent variable and time is the dependent variable.

The principal processes of calculus are differentiation and integration.

150. Differentiation

Differentiation is the process of finding the rate of change of one variable with respect to another. If x is an independent variable, y is a dependent variable, and y is a function of x, this relationship may be written $y = f(x)$. Since for every value of x there is a corresponding value of y, the relationship can be plotted as a curve, Figure 150
. In this figure, *A* and *B* are any two points on the curve, a short distance apart.

The difference between the value of x at *A* and at *B* is Δx (delta x), and the corresponding difference in the value of y is Δ y (delta y). The straight line through points *A* and *B* is a **secant** of the curve. It represents the rate of change between *A* and *B* for anywhere along this line the change of

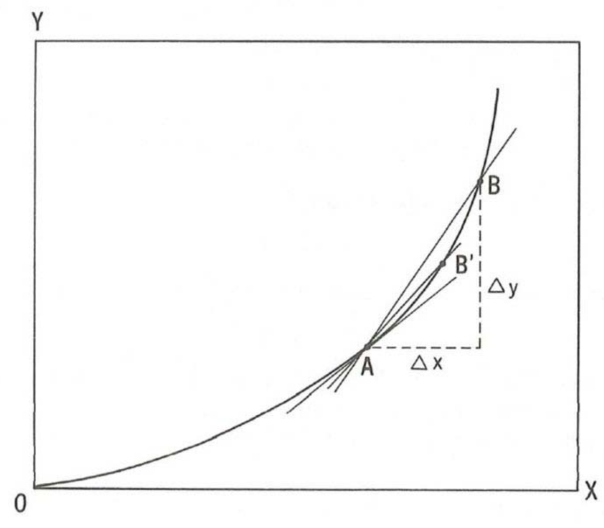

Figure 150. Differentiation.

y is proportional to the change of x.

As *B* moves closer to *A*, as shown at *B'*, both Δ x and Δ y become smaller, but at a different rate, and $\frac{\Delta y}{\Delta x}$ changes. This is indicated by the difference in the slope of the secant. Also, that part of the secant between *A* and *B* moves closer to the curve and becomes a better approximation of it. The limiting case occurs when B reaches A or is at an infinitesimal distance from it. As the distance becomes infinitesimal, both Δ y and Δ x become infinitely small, and are designated *dy* and *dx*, respectively. The straight line becomes tangent to the curve, and represents the rate of change, or slope, of the curve at that point. This is indicated by the expression $\frac{dy}{dx}$, called the **derivative** of y with respect to x.

The process of finding the value of the derivative is called **differentiation**. It depends upon the ability to connect x and y by an equation. For instance, if $y = x^n$, $\frac{dy}{dx} = nx^{n-1}$. If $n = 2$, $y = x^2$, and $\frac{dy}{dx} = 2x$. This is derived as follows: If point *A* on the curve is *x*, *y*; point *B* can be considered $x + \Delta x$, $y + \Delta y$. Since the relation $y = x^2$ is true anywhere on the curve, at *B*:

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + (\Delta x)^2 .$$

Since $y = x^2$, and equal quantities can be subtracted from both sides of an equation without destroying the equality:

$$\Delta y = 2x\Delta x + (\Delta x)^2 .$$

Dividing by Δx:

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x .$$

As *B* approaches *A*, Δx becomes infinitesimally small, approaching 0 as a limit, Therefore $\frac{\Delta y}{\Delta x}$ approached $2x$ as a limit.

This can be demonstrated by means of a numerical example. Let $y = x^2$. Suppose at *A*, $x = 2$ and $y = 4$, and at *B*, $x = 2.1$ and $y = 4.41$. In this case $\Delta x = 0.1$ and $\Delta y = 0.41$, and

$$\frac{\Delta y}{\Delta x} = \frac{0.41}{0.1} = 4.1 .$$

From the other side of the equation:

$$2x + \Delta x = 2 \quad 2 + 0.1 = 4.1 .$$

If Δx is 0.01 and Δy is 0.0401, $\frac{\Delta y}{\Delta x} = 4.01$. If Δx is 0.001, $\frac{\Delta y}{\Delta x} = 4.001$; and if Δx is 0.0001, $\frac{\Delta y}{\Delta x} = 4.0001$. As Δx approaches 0 as a limit, $\frac{\Delta y}{\Delta x}$ approaches 4, which is therefore the value $\frac{dy}{dx}$. Therefore, at point *A* the *rate* of change of *y* with respect to *x* is 4, or *y* is increasing in value 4 times as fast as *x*.

151. Integration

Integration is the inverse of differentiation. Unlike the latter, however, it is not a direct process, but involves the recognition of a mathematical expression as the differential of a known function. The function sought is the **integral** of the given expression. Most functions can be differentiated, but many cannot be integrated.

Integration can be considered the summation of an infinite number of infinitesimally small quantities, between specified limits. Consider, for instance, the problem of finding an area below a specified part of a curve for which a mathematical expression can be written. Suppose it is desired to find the area *ABCD* of Figure 151. If vertical lines are drawn dividing the area into a number of vertical strips, each Δx wide, and if *y* is the height of each strip at the midpoint of Δx, the area of each strip is approximately $y\Delta x$; and the approximate total area of all strips is the sum of the areas of the individual strips. This may be written $\sum_{x1}^{x2} y\Delta x$, meaning the sum of all $y\ \Delta x$ values between x_1 and x_2. The symbol Σ is the Greek letter *sigma*, the equivalent of the English *S*. If Δx is made progressively smaller, the sum of the small areas becomes ever closer to the true total area. If Δx becomes infinitely small, the summation expression is written $\int_{x1}^{x2} ydx$, the symbol *dx* denoting an infinitely small Δx. The symbol $\int$, called the "integral sign," is a distorted *S*.

An expression such as $\int_{x1}^{x2} ydx$ is called a **definite integral** because limits are specified (x_1 and x_2). If limits are not specified, as in $\int ydx$, the expression is called an **indefinite integral**.

A navigational application of integration is the finding of meridional parts, Table 6. The *rate* of change of meridional parts with respect to latitude changes progressively. The formula given in the explanation of the table is the equivalent of an integral representing the sum of the meridional parts from the equator to any given latitude.

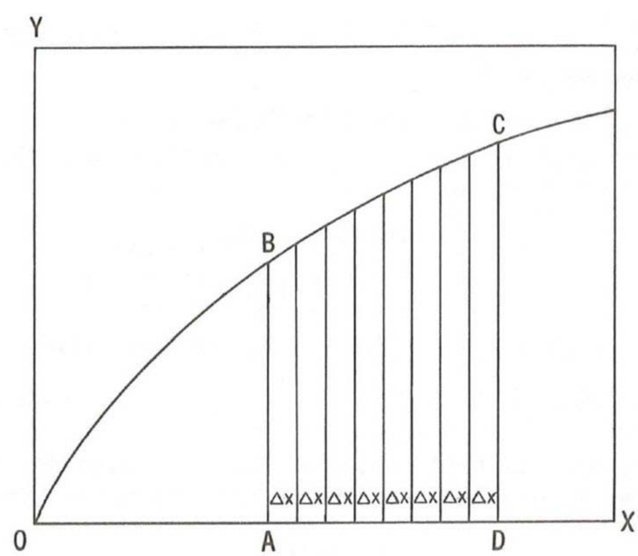

Figure 151. Integration.

152. Differential Equations

An expression such as *dy* or *dx* is called a **differential**. An equation involving a differential or a derivative is called a **differential equation**.

As shown in Section 150, if $y = x^2$, $\frac{dy}{dx} = 2x$. Neither *dy* nor *dx* is a finite quantity, but both are limits to which Δy and Δx approach as they are made progressively smaller. Therefore $\frac{dy}{dx}$ is merely a ratio, the limiting value of $\frac{\Delta y}{\Delta x}$, and not one finite number divided by another. However, since the ratio is the same as would be obtained by using finite quantities, it is possible to use the two differentials *dy* and *dx* independently in certain relationships. Differential equations involve such relationships.

Other examples of differential equations are:

$d \sin x = \cos x \, dx \quad d \csc x = - \cot x \csc x \, dx$

$d \cos x = - \sin x \, dx \quad d \sec x = \tan x \sec x \, dx$

$d \tan x = \sec^2 x \, dx \quad d \cot x = - \csc^2 x \, dx.$

Some differential equations indicating the variations in the astronomical triangle are:

$dh = - \cos L \sin Z \, dt$; *L* and *d* constant

$dh = \cos Z \, dL$; *d* and *t* constant

$dh = - \cos h \tan M \, dZ$; *L* and *d* constant

$dZ = - \sec L \cot t \, dL$; *d* and *h* constant

$dZ = \tan h \sin Z \, dL$; *d* and *t* constant

$dt = - \sec L \cot Z \, dL$; *d* and *h* constant

$dZ = \cos d \sec h \cos M \, dt$; *L* and *d* constant

$dd = \cos d \tan M \, dt$; *L* and *h* constant

$dd = \cos L \sin t \, dZ$; *L* and *h* constant,

where *h* is the altitude, *L* is the latitude, *Z* is the azimuth angle, *d* is the declination, *t* is the meridian angle, and *M* is the parallactic angle.